

# Pattern avoidance in alternating sign rectangles

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# Overview

Pattern avoidance in alternating sign matrices

Classical pattern avoidance in alternating sign rectangles

Extended pattern avoidance in alternating sign rectangles

# Pattern avoidance in alternating sign matrices

# Alternating sign matrices

Alternating sign matrices are a natural generalisation of permutation matrices.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

## Definition

An *alternating sign matrix (ASM)* is a square matrix with entries from  $\{-1, 0, 1\}$  such that

- ▶ the entries in each row and in each column sum to 1;
- ▶ the nonzero entries alternate along each row and each column.

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- ▶ the entries in each row and in each column sum to 1;
- ▶ the nonzero entries alternate along each row and each column.

Each permutation matrix is an ASM without  $-1$ s.

# Pattern avoiding alternating sign matrices

Johansson and Linusson studied pattern avoidance in ASMs.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The ASM (classically) contains the pattern 312.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The ASM (classically) avoids the pattern 132.

**Theorem** (Johansson, Linusson 2007)

There are two Wilf classes of pattern avoidance in ASMs for patterns of length 3:

- ▶ 132-avoiding ASMs are counted by large Schröder numbers.
- ▶ 123-avoiding ASMs are counted by 1, 2, 6, 23, 103, 514, ..., for which no closed formula is known to date.

# Schröder paths

The large Schröder numbers

$$(\text{Sch}_n)_{n \geq 0} = 1, 2, 6, 22, 90, 394, 1806, 8558, \dots$$

count *Schröder paths*: These are subdiagonal lattice paths from  $(0, 0)$  to  $(n, n)$  that consist of steps  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ .

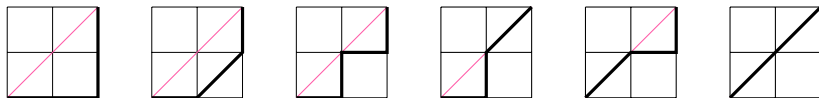


# Schröder paths

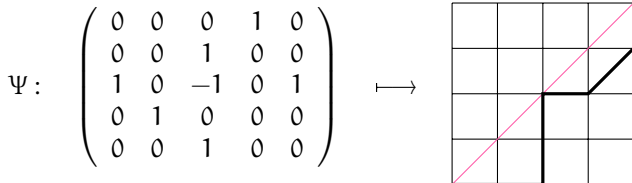
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Johansson, Linusson (2007) constructed an explicit bijection  $\Psi$  between 132-avoiding ASMs of size  $n$  and Schröder paths from  $(0, 0)$  to  $(n - 1, n - 1)$ .



# Stanley–Wilf limit for ASMs

The sequence 1, 2, 6, 23, 103, 514, 2785, 16132, ... of 123-avoiding ASMs is super-exponential. The Stanley–Wilf limit does not exist!

**Theorem** (Bouvel, Egge, Smith, Striker, Troyka 2025)

There exists only a finite class of patterns, namely

$$\{1, 12, 21, 132, 213, 231, 312, 2143, 2413, 3142, 3412\},$$

for which the numbers of pattern avoiding ASMs grow at most exponentially.

**Theorem** (Bouvel, Egge, Smith, Striker, Troyka 2025)

If one fixes the number of  $-1$ s in each ASM, then the numbers of pattern avoiding ASMs grow at most exponentially for every pattern.

Classical pattern avoidance in alternating  
sign rectangles

# Alternating sign rectangles

## Definition

An *alternating sign rectangle (ASR)* is a matrix with entries  $\{-1, 0, 1\}$ , where we call rows and columns without nonzero entries *empty*. We require that

- ▶ the entries in each row sum to 1;
- ▶ the topmost nonzero entry in each nonempty column is a 1;
- ▶ the nonzero entries alternate along rows and columns.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

Let  $a_{r,k}^{\pi}$  denote the number of  $\pi$ -avoiding ASRs of size  $r \times k$ . For the time being, let  $r \leq k$ .

# Wilf classes for pattern avoidance in ASRs

There are three Wilf classes of patterns of length 3:

$$a_{r,k}^{213} = a_{r,k}^{231}:$$

| $r \backslash k$ | 3 | 4  | 5   | 6   |
|------------------|---|----|-----|-----|
| 3                | 6 | 32 | 102 | 252 |
| 4                |   | 22 | 161 | 666 |
| 5                |   |    | 90  | 844 |

$$a_{r,k}^{132} = a_{r,k}^{312}:$$

| $r \backslash k$ | 3 | 4  | 5   | 6    |
|------------------|---|----|-----|------|
| 3                | 6 | 35 | 121 | 322  |
| 4                |   | 22 | 199 | 993  |
| 5                |   |    | 90  | 1179 |

$$a_{r,k}^{123} = a_{r,k}^{321}:$$

| $r \backslash k$ | 3 | 4  | 5   | 6    |
|------------------|---|----|-----|------|
| 3                | 6 | 33 | 108 | 273  |
| 4                |   | 23 | 180 | 790  |
| 5                |   |    | 103 | 1069 |

# Fundamental ASRs

An ASR is *fundamental* if it does not contain empty columns. Let  $f_{r,k}^\pi$  denote the number of fundamental  $\pi$ -avoiding ASRs of size  $r \times k$ .

## Theorem

$a_{r,k}^\pi$  is a polynomial in  $k$ . In particular,

$$a_{r,k}^\pi = \sum_{m=r}^{\binom{r+1}{2}} \binom{k}{m} f_{r,m}^\pi.$$

**Proof idea:** Deleting empty columns from a  $\pi$ -avoiding ASR of size  $r \times k$  with  $m$  non-empty columns is a  $\binom{k}{m}$ -to-1 map. □

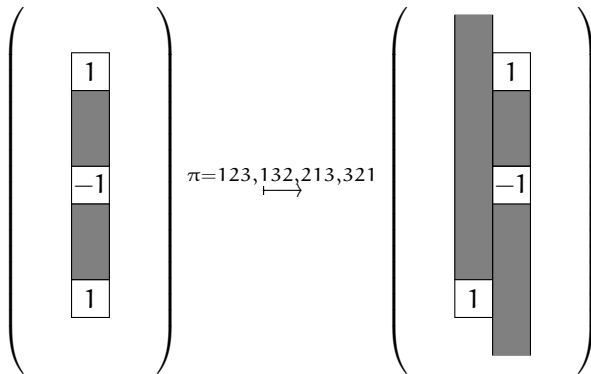
In order to study the asymptotics of  $a_{r,k}^\pi$  for  $k \rightarrow \infty$ , we need to count *maximal*  $\pi$ -avoiding ASRs. We call a fundamental  $\pi$ -avoiding ASR of size  $r \times k$  maximal if there are no fundamental  $\pi$ -avoiding ASRs with  $r$  rows and more than  $k$  columns.

# Maximizing the number of columns

## Lemma

If  $\pi \in \mathfrak{S}_3$ , then every column of a maximal  $\pi$ -avoiding ASR contains exactly one 1.

**Proof idea:**



# Asymptotics of pattern avoiding ASRs I

Let  $f_r^\pi$  denote the number of maximal  $\pi$ -avoiding ASRs with  $r$  rows.  
Let  $d_r^\pi$  the number of columns of a maximal  $\pi$ -avoiding ASR with  $r$  rows.

We then have

$$a_{r,k}^\pi \sim \frac{f_r^\pi}{d_r^\pi!} k^{d_r^\pi}.$$

## Theorem

- ▶ For  $r \geq 1$ , we have  $d_r^{213} = 2r - 1$  and  $f_r^{213} = C_{r-1}$ , where  $(C_r)_{r \geq 0}$  is the sequence of Catalan numbers satisfying the recursion

$$C_{r+1} = \sum_{j=0}^r C_j C_{r-j}.$$

- ▶ ...

# Asymptotics of pattern avoiding ASRs II

Let  $f_r^\pi$  denote the number of maximal  $\pi$ -avoiding ASRs with  $r$  rows.  
Let  $d_r^\pi$  the number of columns of a maximal  $\pi$ -avoiding ASR with  $r$  rows.

## Theorem

- ▶ ...
- ▶ For  $r \geq 2$ , we have  $d_r^{132} = 3r - 3$  and  $f_r^{132} = D_{r-2}$ , where  $(D_r)_{r \geq 0}$  is the sequence 1, 1, 3, 14, 85, 621, 5236, ... (A088716) satisfying the recursion

$$D_{r+1} = \sum_{j=0}^r (j+1) D_j D_{r-j}.$$

- ▶ For  $r \geq 1$ , we have  $d_r^{123} = 2r - 1$  and  $f_r^{123} = E_{r-1}$ , where  $(E_r)_{r \geq 0}$  is the sequence 1, 1, 3, 14, 86, 644, 5627, ...

The sequence 1, 1, 3, 14, 86, 644, 5627, ...

## Definition

For a sequence  $(a_1, \dots, a_r)$ , take another copy of this sequence, place a bar over all the numbers, and order the barred numbers so that they are weakly increasing. Now interweave the two sequences so that for each  $i$ , the elements  $i$  and  $\bar{i}$  alternate (starting with  $i$  and ending with  $\bar{i}$ ). Let  $E_r$  be the number of such interweavements over all sequences  $(a_1, \dots, a_r)$  satisfying  $1 \leq a_j \leq j$ .

## Example

For  $r = 3$ , we have:

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|     |                         |                                                                                                                                                                         |
|-----|-------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 111 | $\bar{1}\bar{1}\bar{1}$ | 1 $\bar{1}$ 1 $\bar{1}$ 1 $\bar{1}$                                                                                                                                     |
| 112 | $\bar{1}\bar{1}\bar{2}$ | 1 $\bar{1}$ 1 $\bar{1}$ 2 $\bar{2}$ , 1 $\bar{1}$ 12 $\bar{1}\bar{2}$                                                                                                   |
| 113 | $\bar{1}\bar{1}\bar{3}$ | 1 $\bar{1}$ 1 $\bar{1}$ 3 $\bar{3}$ , 1 $\bar{1}$ 13 $\bar{1}\bar{3}$                                                                                                   |
| 121 | $\bar{1}\bar{1}\bar{2}$ | 1 $\bar{1}$ 21 $\bar{1}\bar{2}$ , 12 $\bar{1}$ 1 $\bar{1}\bar{2}$                                                                                                       |
| 122 | $\bar{1}\bar{2}\bar{2}$ | 1 $\bar{1}$ 2 $\bar{2}$ 2 $\bar{2}$ , 12 $\bar{1}$ 2 $\bar{2}\bar{2}$                                                                                                   |
| 123 | $\bar{1}\bar{2}\bar{3}$ | 1 $\bar{1}$ 2 $\bar{2}$ 3 $\bar{3}$ , 1 $\bar{1}$ 23 $\bar{2}\bar{3}$ , 12 $\bar{1}$ 2 $\bar{3}\bar{3}$ , 12 $\bar{1}$ 3 $\bar{2}\bar{3}$ , 123 $\bar{1}\bar{2}\bar{3}$ |

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Extended pattern avoidance in alternating  
sign rectangles

# Other notions of pattern avoidance/containment

For (square) ASMs:

- zero-nonzero-patterns [Brualdi, Kiernan, Meyer, Schroeder, 2013]
- key avoidance [Bouvel, Smith, Striker 2025]

For partial permutation matrices:

- extended avoidance [Eriksson, Linusson, 1995]

## Definition

A  $\{0, 1\}$ -matrix of size  $r \times k$  with at most one 1 in each row and columns (*partial permutation*) is said to be *extendably  $\pi$ -avoiding* if it can be extended downwards and to the right to a  $\pi$ -avoiding permutation matrix.

# Extended avoidance

A  $\{0, 1\}$ -matrix of size  $r \times k$  with at most one 1 in each row and columns is said to be *extendably  $\pi$ -avoiding* if it can be extended downwards and to the right to a  $\pi$ -avoiding permutation matrix.

## **Theorem** (Linusson 2002)

There are two Wilf classes for extendably  $\pi$ -avoiding permutations for patterns  $\pi$  of length 3, namely  $\{321, 312, 231, 132\}$  and  $\{123, 213\}$ . Both classes can be enumerated with a closed formula. In particular, the former is counted by the *ballot numbers*, which count Dyck paths from  $(0, 0)$  to  $(n, m)$  for  $0 \leq m \leq n$  that stay under the main diagonal.

# Extendable avoidance for ASRs

An ASR of size  $r \times k$  is said to be *extendably  $\pi$ -avoiding* if it can be extended downwards and to the right to a  $\pi$ -avoiding square ASM.

## Example

The following  $4 \times 6$  ASR is extendably 312-avoiding:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

# Extended avoidance for ASRs

Let  $S_{r,k,d}^\pi$  be the number of extendably  $\pi$ -avoiding ASRs of size  $r \times k$  with  $d$  nonempty rows.

- ▶  $S_{r,k,d}^\pi = 0$  if  $r < d$  or  $k < d$ .
- ▶ If  $d = r$ , there are no empty rows; if  $d = k$ , there are no empty columns.
- ▶  $S_{n,n,n}^\pi = \text{Sch}_{n-1}$  if  $\pi \in \{312, 132, 213, 231\}$ .

## Theorem

We have a complete recursive description of the numbers  $(S_{r,k,d}^\pi)_{r,k \geq d \geq 0}$  of extendably  $\pi$ -avoiding ASR of size  $r \times k$  with  $d$  nonempty rows if  $\pi \in \{312, 132, 213, 231\}$ .

# Key observation

## Theorem

There cannot be more than one  $-1$  in any row or column of an extendably  $\pi$ -avoiding ASR for any  $\pi \in \mathfrak{S}_3$ .

**Proof idea for  $\pi = 312$ :**

$$\begin{pmatrix} & & & 1 & \\ 1 & -1 & 1 & -1 & 1 \\ & 1 & & & \end{pmatrix}$$

$$\begin{pmatrix} & 1 & & & \\ & -1 & 1 & & \\ & 1 & & & \\ 1 & & -1 & & \\ & & & 1 & \end{pmatrix}$$

# The recursion for $\pi = 312$

- $d = r$ : no empty rows

$$S_{r,k,r}^{312} = S_{r,k-1,r}^{312} + S_{r-1,k-1,r}^{312} + \sum_{i=2}^r \text{Sch}_{i-2} \cdot S_{r+1-i,k+1-i,r+1-i}^{312}.$$

- $d = k$ : no empty columns

$$S_{r,k,k}^{312} = \sum_{j=0}^k S_{j,k,j}^{312} \binom{r-j-1}{k-j}.$$

- $d < r, k$ : the general case

$$\begin{aligned} S_{r,k,d}^{312} &= S_{r,k-1,d}^{312} + S_{r-1,k-1,d-1}^{312} \\ &\quad + \sum_{i=2}^d \text{Sch}_{i-2} \cdot \left( S_{r+1-i,k+1-i,d+1-i}^{312} - \binom{r-i}{d+1-i} \right). \end{aligned}$$

Together with the base cases  $S_{0,k,0}^{312} = 1 = S_{r,0,0}^{312}$  and  $S_{n,n,n}^{312} = S_{n-1}$ , we have a complete recursive description of  $S_{r,k,d}^{312}$  for any  $r, k \geq d \geq 0$ .

# Combinatorial observations for $\pi = 312$

- For  $2 \leq r \leq k$ , we have  $S_{r,k,r}^{312} = \sum_{j=2}^r \text{Sch}_{r-1,j} \binom{k-r+j}{j}$ , where  $\text{Sch}_{i,j}$  denotes the number of Schröder paths from  $(0,0)$  to  $(i,i)$  that touch the diagonal exactly  $j$  times.
- For  $1 \leq r \leq k$ ,  $S_{r,k,r}^{312} - S_{r,k-1,r}^{312}$  equals the number of Schröder paths from  $(0,0)$  to  $(k-1, r-1)$  (*Schröder ballot numbers*).
- The sequence  $(S_{r,r+2,r}^{312})_{r \geq 0}$  counts *Delannoy paths*, that is, paths with steps  $(1,0)$ ,  $(0,1)$  and  $(1,1)$ , that do not cross the diagonal  $y = x$  vertically (A268208).  $\rightsquigarrow$  bijection
- We can write  $S_{r,k,k}^{312} = \sum_{j=0}^k c_{j,k} \binom{r-j}{k-j}$ , where  $c_{j,k} = S_{j,k,j}^{312} - S_{j-1,k,j-1}^{312}$ . The numbers  $c_{j,k}$  are a refinement of the large Schröder numbers. What do they count?

| $k \backslash j$ | 0 | 1 | 2  | 3  | 4  | $\Sigma$ |
|------------------|---|---|----|----|----|----------|
| 1                | 1 |   |    |    |    | 1        |
| 2                | 1 | 1 |    |    |    | 2        |
| 3                | 1 | 2 | 3  |    |    | 6        |
| 4                | 1 | 3 | 8  | 10 |    | 22       |
| 5                | 1 | 4 | 15 | 32 | 38 | 90       |

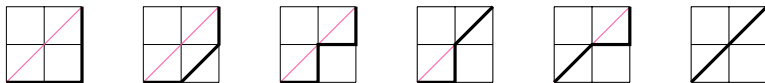
# Open questions

- Find a formula for the sequence

$$(E_r)_{r \geq 0} = 1, 1, 3, 14, 86, 644, 5627, \dots$$

- What do the numbers  $c_{j,k}$  count?

Find a statistic on these lattice paths with distribution  $(1, 2, 3)$ :



- We conjecture that

$$S_{r,k,d}^{312} = \sum_{i=1}^d 2^{i-1} \binom{k}{i} \left( \binom{r-2}{d-i} - 2 \binom{r-2}{d-i-2} \right) + \binom{r}{d} \\ + \sum_{i \geq 0} c_{2i} \binom{r-1+2i}{d-1-2i} + 2 \sum_{i=1}^d (-1)^i \binom{r}{d-i}.$$

Thank you!