

A Graph on Bumpless Pipe Dreams

Joshua Arroyo

joint work with Adam Gregory and Prashreet Poudel

July 24, 2026

Bumpless Pipe Dreams

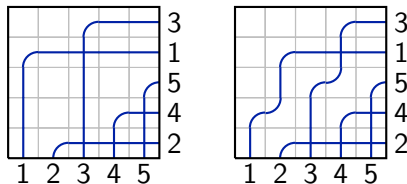
Definition (Bumpless Pipe Dream)

A *bumpless pipe dream (BPD)* is a $n \times n$ filling using the tiles



such that pipes enter from the bottom and exit out of the right and no pair of pipes cross more than once.

Each bumpless pipe dream has an associated permutation given by the order of its pipes.



Schubert Polynomials

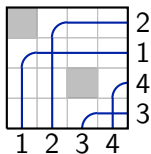
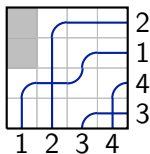
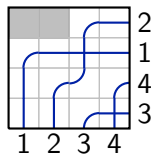
Definition (Schubert Polynomial)

A Schubert polynomial of a permutation w is

$$\mathfrak{S}_w = \sum_{B \in \text{bpd}(w)} wt(B),$$

where $\text{bpd}(w)$ is the set of all BPDs of w and each empty box in row i has weight x_i .

Schubert Polynomials



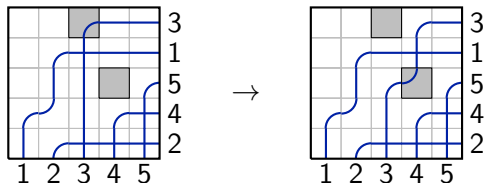
$$\mathfrak{S}_{2143} = x_1x_3 + x_1x_2 + x_1^2$$

$$\mathfrak{S}_{2143}(1, 1, 1) = 3$$

Droop Move

Definition

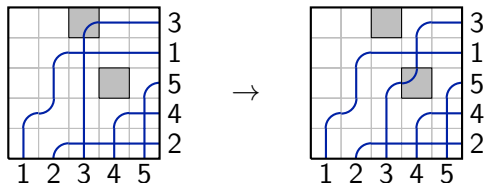
A droop move on a BPD is a map between two BPDs of the same permutation made by "drooping" an  tile into a  tile.



Droop Move

Definition

A droop move on a BPD is a map between two BPDs of the same permutation made by "drooping" an  tile into a  tile.



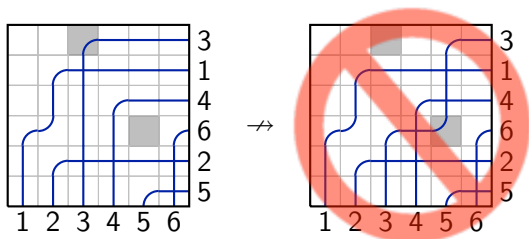
Note

As pipes cannot double cross there cannot be any elbow tiles between the chosen  and  tiles.

Droop Move

Definition

A droop move on a BPD is a map between two BPDs of the same permutation made by "drooping" an  tile into a  tile.



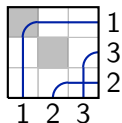
Note

As pipes cannot double cross there cannot be any elbow tiles between the chosen  and  tiles.

132 Patterns

Observation

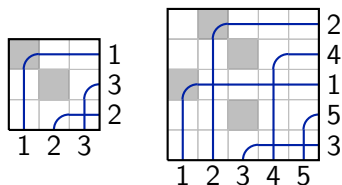
Droop moves correspond with 132 patterns.



132 Patterns

Observation

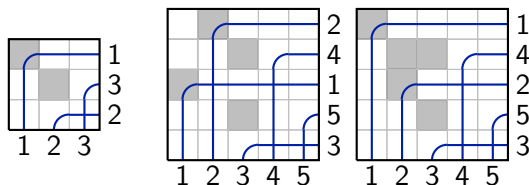
Droop moves correspond with 132 patterns.



132 Patterns

Observation

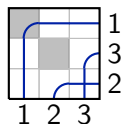
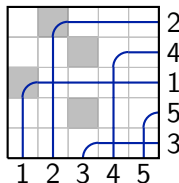
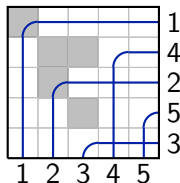
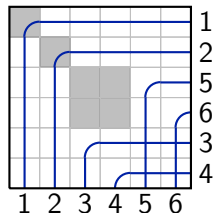
Droop moves correspond with 132 patterns.



132 Patterns

Observation

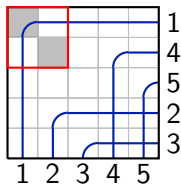
Droop moves correspond with 132 patterns.



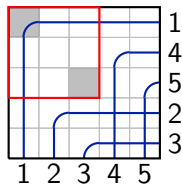
Min Droop Move

Definition

A min droop move is a droop move where there are no other  tiles between the chosen  and  tiles.



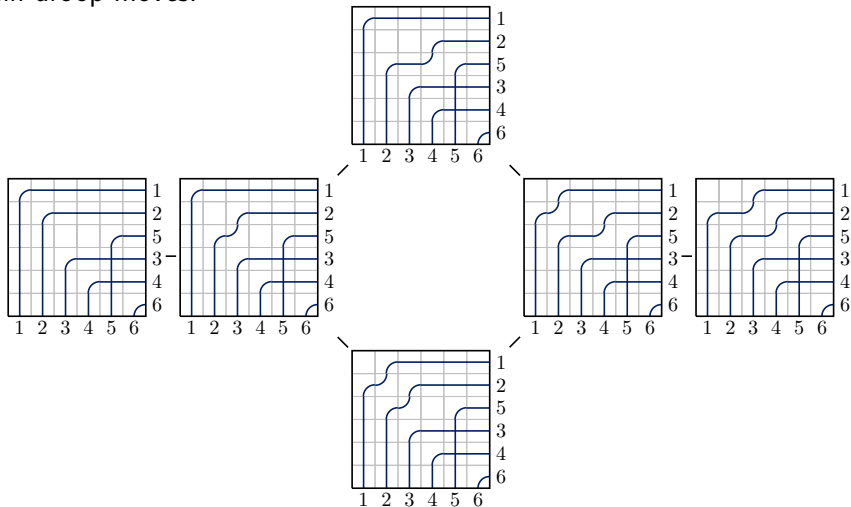
Min Droop



Not Min Droop

Droop Graph

All bumpless pipe dreams of a given permutation are connected by min droop moves.

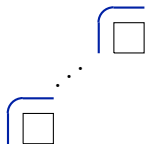


Theorem (A, Gregory, Poudel 2026+)

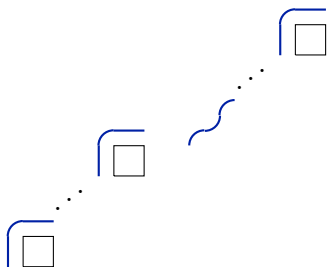
The BPD droop graph of a permutation w has a cycle if and only if w contains at least one of the following patterns:

- 13254, 13524, 14253, 24153, 31524, 35142, 354162, 461325, 465132,
- 1432, 14523,
- 12534, 12453,
- 214563, 216345

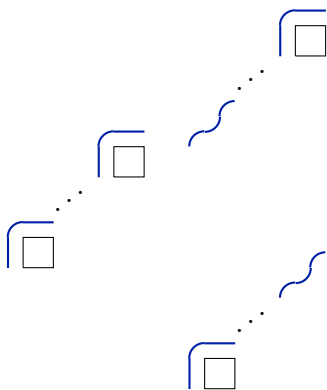
Case 1



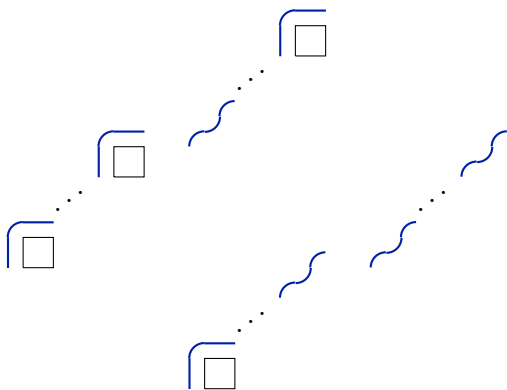
Case 1



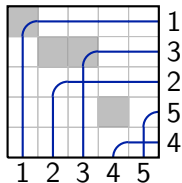
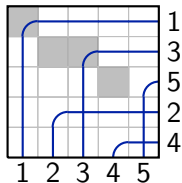
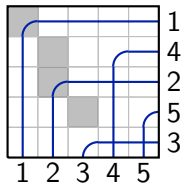
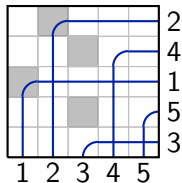
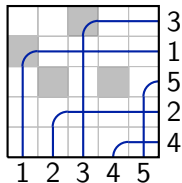
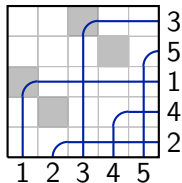
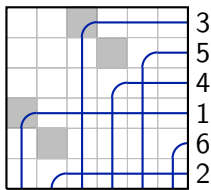
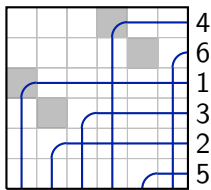
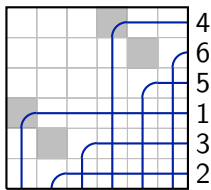
Case 1



Case 1



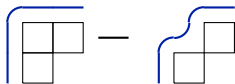
Case 1



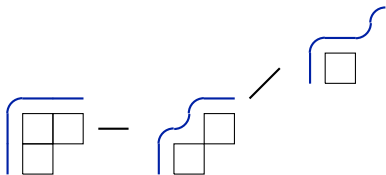
Case 2



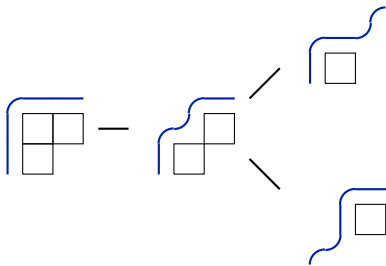
Case 2



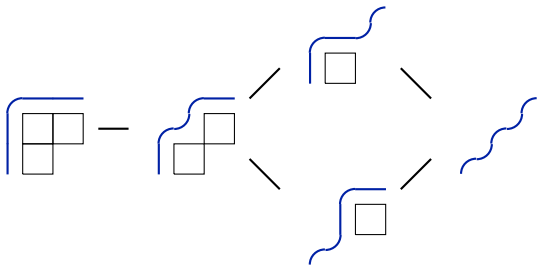
Case 2



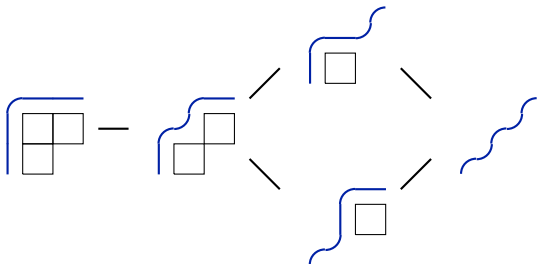
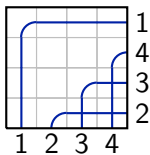
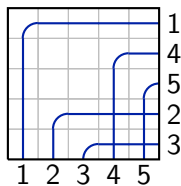
Case 2



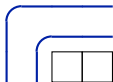
Case 2



Case 2



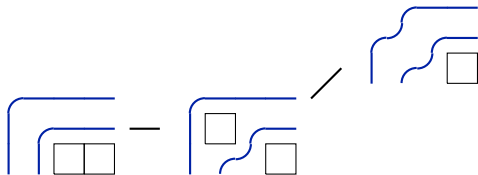
Case 3



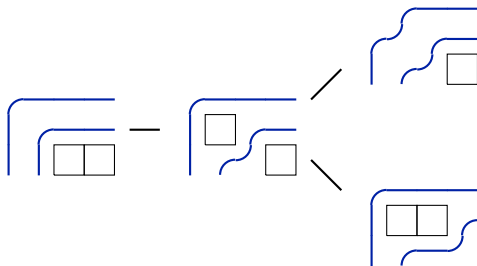
Case 3



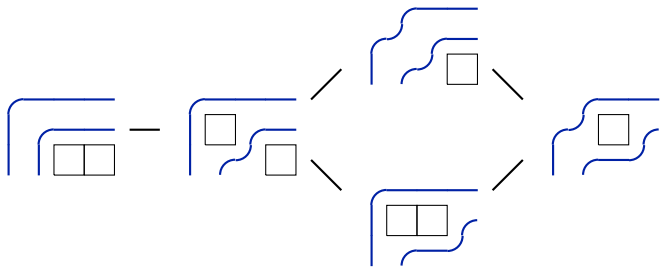
Case 3



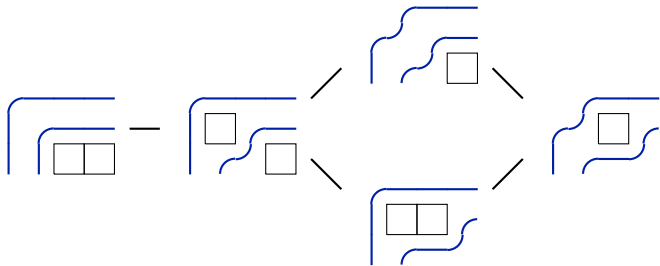
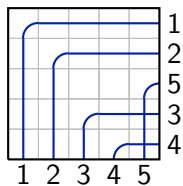
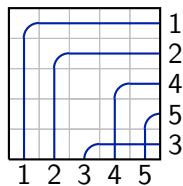
Case 3



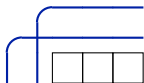
Case 3



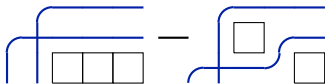
Case 3



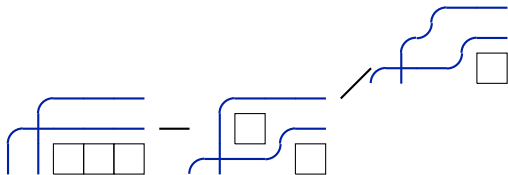
Case 4



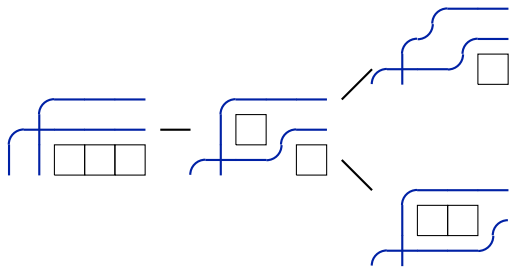
Case 4



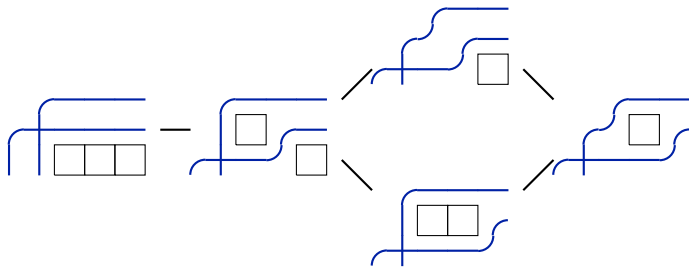
Case 4



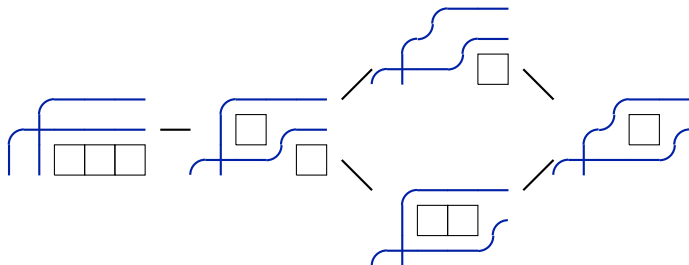
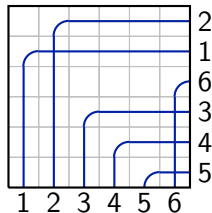
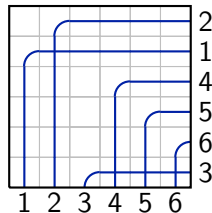
Case 4



Case 4



Case 4



Theorem (A, Gregory, Poudel 2026+)

The BPD graph of a permutation w is a cycle if and only if w contains at least one of the following permutations:

- 13254, 13524, 14253, 24153, 31524, 35142, 354162, 461325, 465132,
- 1432, 14523,
- 12534, 12453,
- 214563, 216345

Thanks

Thank you for listening.