

The Rtlm sorting algorithm

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Preliminaries

Definition

Define a *barred pattern* as a permutation that has bars above some of its entries. We then say a permutation π *avoids a barred pattern* τ if every occurrence of the non-barred entries of τ in π extends to a full occurrence of τ .

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Example

The permutation $\pi = 6\bar{3}5\bar{1}42$ contains the pattern $\bar{1}324$ because π contains the subsequence 314 that matches the pattern 324, but does not extend to a 1324 pattern.

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Example

The permutation $\pi = 6\bar{3}5\bar{1}42$ contains the pattern $\bar{1}324$ because π contains the subsequence 314 that matches the pattern 324, but does not extend to a $\bar{1}324$ pattern.

However, π avoids the pattern $\bar{4}213$ because the only subsequence of π matching the unbarred 213 pattern extends to a 4213 pattern, specifically the subsequence 6314.

Definition

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Example

Notice $\pi = 635142$ avoids 213 as a contiguous pattern.

However, π contains the contiguous pattern 132 because the entries 354 are consecutive in value and form a 132 pattern.

Definition

Given $\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n$, a *right-to-left minima* is an entry π_i such that $\pi_i < \pi_j$ for $i < j \leq n$.

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Example

The permutation $\pi = 253194687$ has right-to-left minima 1, 4, 6, 7.

The Rtlm sorting algorithm

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 $R'(\pi) = \{\pi_1, \dots, \pi_n\} \setminus R(\pi)$.

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Define $S_{rlm}(\pi)$ to be the permutation obtained by listing all entries of $R(\pi)$ followed by all entries of $R'(\pi)$ where each set of entries appears in the same order as they appeared in π .

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Example

Let $\pi = 243561$.

input	set of right-to-left minima	output
$\pi = 243561$	$R(\pi) = \{1\}$	$S_{rlm}(\pi) = 124356$
$S_{rlm}(\pi) = 124356$	$R(\pi) = \{1, 2, 3, 5, 6\}$	$S_{rlm}^2(\pi) = 123564$
$S_{rlm}^2(\pi) = 123564$	$R(\pi) = \{1, 2, 3, 4\}$	$S_{rlm}^3(\pi) = 123456$

Theorem

For all $\pi \in \mathcal{S}_n$, π is sortable after at most $n - 1$ applications of S_{rlm} . Further, $n - 1$ is a sharp upper bound.

Some numbers

Let $\mathcal{T}(n, k)$ be the set of permutations of size n sortable via at most k applications of S_{rlm} .

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Let $T^*(n, k)$ be the set of permutations of size n sortable via exactly k applications of S_{rlm} .

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Let $T^*(n, k)$ be the set of permutations of size n sortable via exactly k applications of S_{rlm} .

Further, let $t(n, k) = |T(n, k)|$ and let $t^*(n, k) = |T^*(n, k)|$.

$n \backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0
4	1	7	11	5	0	0	0	0
5	1	15	43	45	16	0	0	0
6	1	31	148	268	211	61	0	0
7	1	63	480	1344	1767	1113	272	0
8	1	127	1509	6171	12099	12477	6551	1385

Table 1: Number of permutations of size n requiring exactly k applications of S_{rlm} to be in increasing order.

$n \backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0
4	1	7	11	5	0	0	0	0
5	1	15	43	45	16	0	0	0
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Table 1: Number of permutations of size n requiring exactly k applications of S_{rlm} to be in increasing order.

Fact

$$t(n, 0) = 1.$$

$n \backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0
4	1	7	11	5	0	0	0	0
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Theorem

For $n \geq 0$, $T(n, 1) = \mathcal{S}_n(321, 314\bar{2})$, and moreover $t(n, 1) = 2^{n-1}$.

$n \backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0
4	1	7	11	5	0	0	0	0
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Table 1: Number of permutations of size n requiring exactly k applications of S_{rlm} to be in increasing order.

Theorem

For all $n \geq 0$, we have $t(n, 2) = \frac{3^n - 2n + 3}{4}$.

$n \backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0
4	1	7	11	5	0	0	0	0
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Theorem

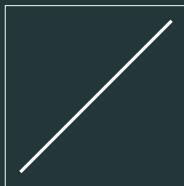
For all $n \geq 0$, $t(n, 3) = 2^{2n-3} - (n-2)2^{n-2} = \frac{4^n + 2^{n+1}(2-n)}{8}$.

Some pictures

What do permutations in $T^*(n, k)$ look like?

Note: To obtain a permutation in $T^*(n, k)$, we have an additional layer of right-to-left minima that has k sets of consecutive values, meant to interleave nicely with the increasing runs at the start of $T(n, k - 1)$.

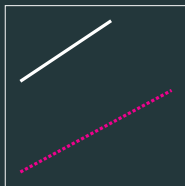
Permutations in $T^*(n, 0)$ look like:



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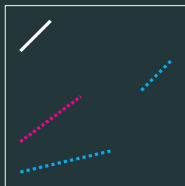
Permutations in $T^*(n, 1)$ look like:



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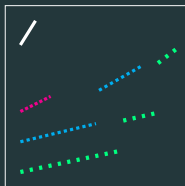
Permutations in $T^*(n, 2)$ look like:



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Permutations in $T^*(n, 3)$ look like:



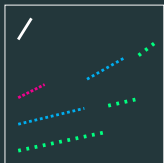


Figure 1: A member of $T(n, 3)$ being sorted.

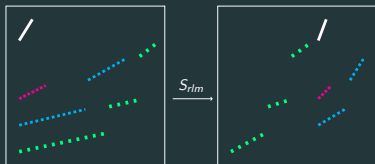


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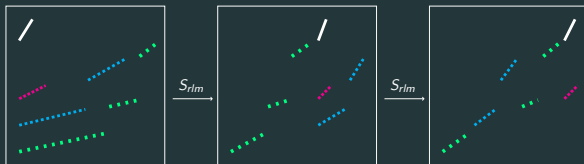


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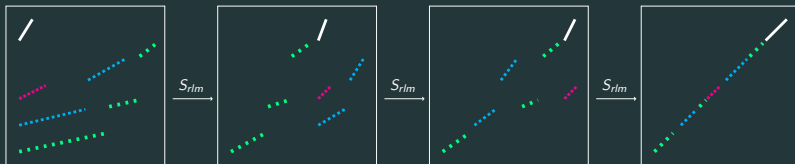


Figure 1: A member of $T(n, 3)$ being sorted.

Some bounds

Recall, we have

$$t(n, 0) = \frac{1^n}{1},$$

$$t(n, 1) = \frac{2^n}{2},$$

$$t(n, 2) = \frac{3^n - 2n + 3}{4},$$

$$t(n, 3) = \frac{4^n + 2^{n+1}(2 - n)}{8}.$$

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Theorem

Suppose $k \geq 0$ is fixed, then

$$t(n, k) \leq \frac{(k+1)^n}{2^k} \text{ ish.}$$

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Theorem

Suppose $k \geq 0$ is fixed, then

$$\frac{(k+1)^n}{(k+1)^k} \leq t(n, k) < |\text{Grid}([1 \cdots 1])| \text{ (skinny grid class with } k+1 \text{ cells)}$$

$$\text{and } |\text{Grid}([1 \cdots 1])| \sim \frac{(k+1)^n}{2^k} \text{ (Alshammari and Bevan, 2026)}.$$

Worst case scenarios

Theorem

Let $\pi \in \mathcal{S}_n$ where $n \geq 2$. Then $\pi \in T^(n, n-1)$ if and only if π avoids the contiguous patterns 123, 231, and 312 and is such that the 2 appears before the 1 in π .*

Theorem

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That is, $\pi \in T^(n, n-1)$ if and only if $1 \oplus \pi$ avoids the contiguous patterns 123, 231, and 312.*

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($\rightarrow$) Proof sketch:

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($\rightarrow$) Proof sketch:

Lemma

Suppose $S_{rlm}^k(\pi)$ contains a 123, 231, or 312 pattern in consecutive values. Then $S_{rlm}^{k+1}(\pi)$ also contains a 123, 231, or 312 pattern involving the same entries.

Theorem

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($\rightarrow$) Proof sketch:

Lemma

Suppose $S_{rlm}^k(\pi)$ contains a 123, 231, or 312 pattern in consecutive values. Then $S_{rlm}^{k+1}(\pi)$ also contains a 123, 231, or 312 pattern involving the same entries.

Hence if $1 \oplus \pi$ contained one of the contiguous patterns 123, 231, or 312, then at least two entries will be put in their proper position in the same step.

Euler up/down and Entringer numbers

$n \backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0
4	1	7	11	5	0	0	0	0
5	1	15	43	45	16	0	0	0
6	1	31	148	268	211	61	0	0
7	1	63	480	1344	1767	1113	272	0
8	1	127	1509	6171	12099	12477	6551	1385

Table 2: Number of permutations of size n requiring exactly k applications of S_{rlm} to be in increasing order.

Theorem

The permutations in $T^(n, n-1)$ are enumerated by the Euler up/down numbers for $n \geq 1$.*

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The permutations in $T^(n, n-1)$ are enumerated by the Euler up/down numbers for $n \geq 1$.*

$T^*(n, n-1)$ and the Euler up-down permutations can be described by a generating tree with

Root: $(1,1)$

Rule: $(k, n) \rightarrow (n-k+2, n+1)(n-k+3, n+1) \cdots (n+1, n+1)$.

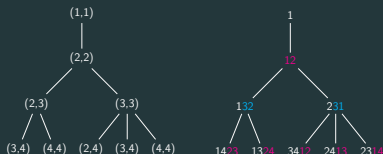


Figure 2: The first four levels of the generating tree for up-down permutations as labels (on left) and as permutations (on right).

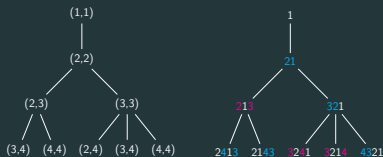


Figure 3: The first four levels of the generating tree for $T^*(n, n-1)$ as labels (on left) and as permutations (on right).

A recursive formula for the Entringer numbers is given by:

$$E_{n,k} = E_{n,k-1} + E_{n-1,n-k},$$

with boundary conditions $E_{0,0} = 1$ and $E_{n,0} = 0$ for $n \geq 1$.

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$n \backslash k$	0	1	2	3	4	5
0	1					
1	0	1				
2	0	1	1			
3	0	1	2	2		
4	0	2	4	5	5	
5	0	5	10	14	16	16

Table 3: Entringer numbers $E_{n,k}$ for small n and k .

Theorem

The permutations in $T^(n, n-1)$ are an Entringer family where $E_{n,k}$ counts the number of permutations in $T^*(n+1, n)$ that have the 1 in position $k+1$.*

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Say $V_{n,k}$ is the set of permutations in $T^*(n+1, n)$ that have the 1 in position $k+1$. Partition $V_{n,k}$ into two subsets:

- $V'_{n,k}$ contains the permutations of $V_{n,k}$ which do not begin with the 2.
- $V''_{n,k}$ contains the permutations of $V_{n,k}$ which do begin with the 2.

The permutations of $V'_{n,k}$ can be rotated so that each entry moves one position to the left except the first entry which moves to the end of the permutation.

The permutations of $V'_{n,k}$ can be rotated so that each entry moves one position to the left except the first entry which moves to the end of the permutation.

The resulting permutation still has the 2 to the left of the 1 and all of the required pattern avoidance because of the rotational symmetry of the avoided patterns 123, 231, 312.

Hence these permutations are elements of $V_{n,k-1}$.

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Hence these permutations are elements of $V_{n,k-1}$.

Example

Let $\pi = 52413 \in V'_{4,3}$. Applying our map, the resulting permutation is $24135 \in V_{4,2}$.

Now consider the permutations of $V''_{n,k}$. As these permutations begin with the 2 and avoid 231 patterns with consecutive values, the 3 of the permutation must appear after the 1 in position $k + 1$. Map the permutations of $V''_{n,k}$ to the permutations of $V_{n-1,n-k}$ as follows.

1. Rotate the permutation k positions to the left.
2. Remove the 1 (currently in position 1) and reduce each remaining entry by one to get a permutation on $[n]$.

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The new 1 is now in position $n - k + 1$, the new 2 appears to the left of that 1, and that again the rotational movement maintains the necessary pattern avoidance. Hence these permutations belong to $V_{n-1,n-k}$.

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The new 1 is now in position $n - k + 1$, the new 2 appears to the left of that 1, and that again the rotational movement maintains the necessary pattern avoidance. Hence these permutations belong to $V_{n-1,n-k}$.

Example

Let $\pi = 25413 \in V''_{4,3}$. Applying our map:
 $25413 \rightarrow 13254 \rightarrow 3254 \rightarrow 2143 \in V_{3,1}$.

Hence,

$$V_{n,k} = V_{n,k-1} + V_{n-1,n-k}.$$

Also note, $V_{0,0} = 1$ and $V_{n,0} = 0$ for $n \geq 1$.

$n \backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0
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Table 4: Number of permutations of size n requiring exactly k applications of S_{rlm} to be in increasing order.

$n \backslash k$	0	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	3	2	0	0	0	0	0
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Table 4: Number of permutations of size n requiring exactly k applications of S_{rlm} to be in increasing order.

Theorem

For $n \geq 2$, we have: $t^*(n+1, n) = t^*(n, n-1) + t^*(n, n-2)$.

Open Questions

1. We know the sequence $\{t(n, 2)\}$ matches OEIS A111277. Are there nice bijections for:

$$t(n, 2) = |\mathcal{S}_n(2413, 4213, 2431, 4231, 4321)| \text{ and}$$

$$t(n, 2) = |\mathcal{S}_n(3142, 3412, 3421, 4312, 4321)|?$$

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2. $\{t(n, 3)\}$ matches OEIS A376585 which has no combinatorial interpretations, but we do have case work that matches OEIS sequences that do and could have nice bijections.

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2. $\{t(n, 3)\}$ matches OEIS A376585 which has no combinatorial interpretations, but we do have case work that matches OEIS sequences that do and could have nice bijections.
3. $\{t(n, k)\}$ grows exponentially for any fixed k . While we know $t(n, k) \sim c(k+1)^n$, it would be interesting to more precisely determine the constant c .

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$$t(n, 2) = |\mathcal{S}_n(3142, 3412, 3421, 4312, 4321)|?$$

2. $\{t(n, 3)\}$ matches OEIS A376585 which has no combinatorial interpretations, but we do have case work that matches OEIS sequences that do and could have nice bijections.
3. $\{t(n, k)\}$ grows exponentially for any fixed k . While we know $t(n, k) \sim c(k+1)^n$, it would be interesting to more precisely determine the constant c .
4. Is there a nice way to characterize the sortable permutations for nonextremal values of k ?

Thank you!

