On the Schur positivity of small descent enumerators for pattern avoiding sets

Kyle Celano

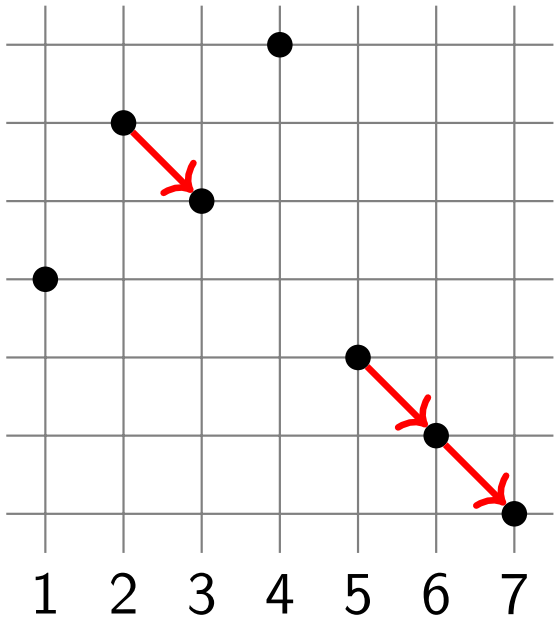
Wake Forest University

Permutation Patterns 4!

Descents and Small Descents

Let $\mathfrak{S}_n := \{\text{permutations } \sigma \text{ of } [n] = \{1, 2, \ldots, n\}\}$.

A small descent of σ is a position $i \in [n-1]$ such that $\sigma_i = \sigma_{i+1} + 1$. Let $\mathrm{SDES}(\sigma)$ be the set of small descents.



$$\mathrm{SDES}(\sigma) = \{2, 5, 6\}$$

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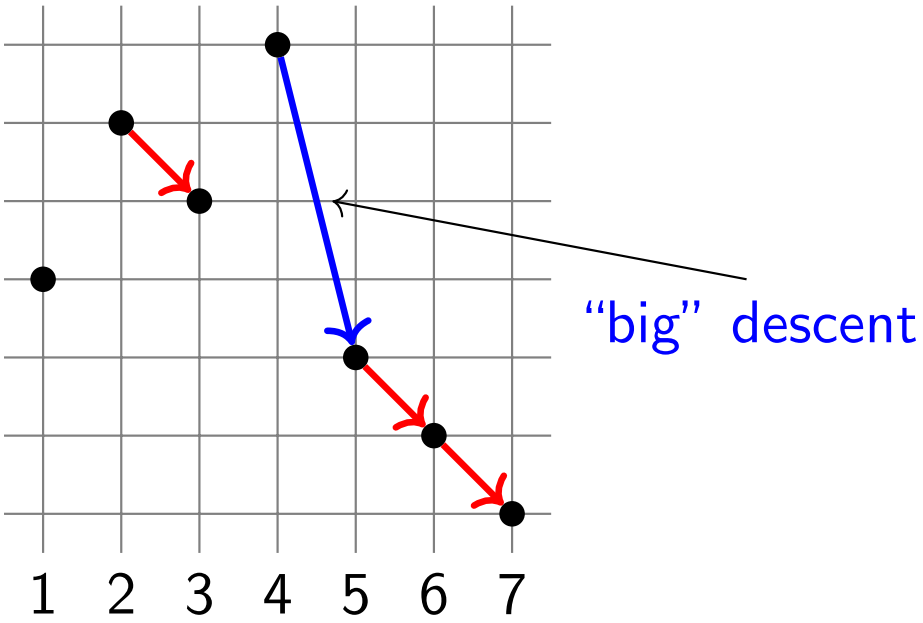

$$\mathrm{SDES}(\sigma) = \{2, 5, 6\}$$

Small descents

Set $\mathrm{sdes}(\sigma) := |\mathrm{SDES}(\sigma)|$

Ex. $n = 4$.

k	$\mathrm{sdes}(\pi) = k$
0	1234, 1342, 1423, 2314, 2341, 2413, 3124, 3142, 3412, 4123, 4231
1	1243, 1324, 2134, 2431, 3241, 3421, 4132, 4213, 4312
2	1432, 2143, 3214
3	4321

Small descents

Ex. $n = 4$. Let's look at $\mathrm{SDES}(\pi) \subseteq [n-1]$ directly

S	$\mathrm{SDES}(\pi) = S$
$\emptyset$	$1234, 1342, 1423, 2314, 2341, 2413,$
	$3124, 3142, 3412, 4123, 4231$
$\{1\}$	$2134, 3241, 4312$
$\{2\}$	$1324, 2431, 4213$
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$|\{\pi \in \mathfrak{S}_n : \mathrm{SDES}(\pi) = S\}|$ seems to only depend on $|S|$

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Proposition

$|\{\pi \in \mathfrak{S}_n : \mathrm{SDES}(\pi) = S\}|$ *only depends on* $|S|$.

Small descents

Ex. $n = 4$. Consider $\mathrm{Av}_4(312)$

S	$\mathrm{SDES}(\pi) = S$
$\emptyset$	$1234, 1342, \cancel{1423}, 2314, 2341, \cancel{2413},$ $\cancel{3124}, \cancel{3142}, \cancel{3412}, \cancel{4123}, \cancel{4231}$
$\{1\}$	$2134, 3241, \cancel{4312}$
$\{2\}$	$1324, 2431, \cancel{4213}$
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Proposition

$|\{\pi \in \mathrm{Av}_n(312) : \mathrm{SDES}(\pi) = S\}|$ *only depends on* $|S|$.

Small descents

Ex. $n = 4$. Consider $\mathrm{Av}_4(312, 123)$

S	$\mathrm{SDES}(\pi) = S$
$\emptyset$	$\emptyset$
$\{1\}$	3241
$\{2\}$	2431
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Small Descents

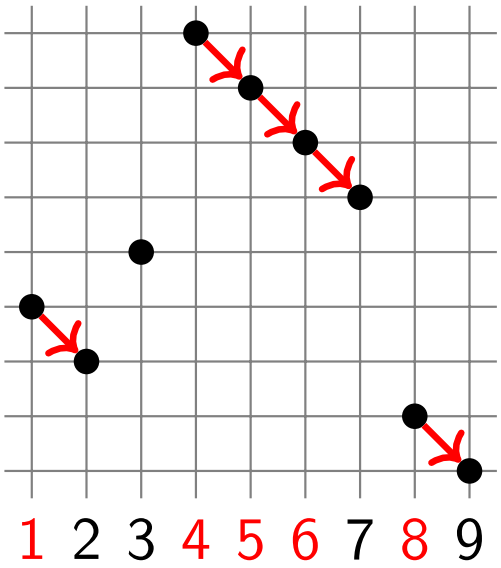
Theorem (C. +26)

Let Π *be a finite set of patterns such that* $\mathrm{SDES}(\pi) = \emptyset$ *for all* $\pi \in \Pi$. *Then, for all* $S \subseteq [n-1]$, $|\{\pi \in \mathrm{Av}_n(\Pi) : \mathrm{SDES}(\pi) = S\}|$ *only depends on* $|S|$.

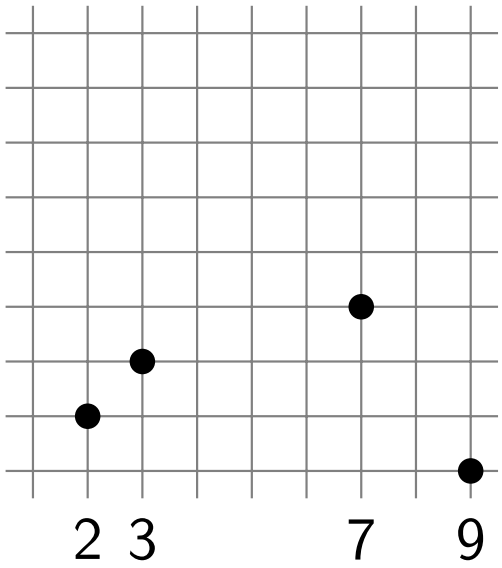
I.E. Any two sets of the same size are equally likely to be small descent sets

Proof

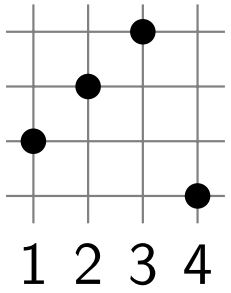
For a given $S \subseteq [n-1]$, one shows a bijection
$\{\sigma \in \mathrm{Av}_n(\Pi) : \mathrm{SDES}(\sigma) = S\} \leftrightarrow \{u \in \mathrm{Av}_{n-|S|}(\Pi) : \mathrm{SDES}(\sigma) = \emptyset\}$



(a) σ



(b) u (spread out)



(c) u

Observe: σ contains small-descent-avoiding pattern iff u does. □

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Q: Why is this an "interesting" property?

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A: Representation theory.

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Punchline: This theorem tells you that there is an action of the symmetric group on $\mathrm{Av}_n(\Pi)$ that "decomposes" via small descents.

Symmetric Functions

Symmetric function: a (homogenous) polynomial f in $x_1, x_2, x_3, \ldots$ that is invariant under any permutation $\sigma : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ of its indices i.e. $f(x_1, x_2, x_3, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \ldots)$

$$2x_1^2x_2 + 2x_1x_2^2 + 2x_1^3x_3 + 2x_1x_3^2 + \cdots \quad \checkmark$$

$$2x_1^2x_2 - 3x_1x_2^2 + 2x_1^3x_3 + 2x_1x_3^2 + \cdots \quad \text{X}$$

Schur functions

Semistandard Young tableaux (SSYT) of shape λ: a filling of a diagram of λ by positive integers that has weakly increasing rows and strictly increasing columns.

1	2	2	8	8
2	3	9	9	21
6	21			
21				

(a) semistandard Young tableau T

$$x^T := \prod_{i \geq 1} x_i^{\#\text{ of } i \text{ in } T}$$

$$x_1 x_2^3 x_3 x_6 x_8^2 x_9^2 x_{21}^3$$

(b) the monomial x^T

Schur function s_λ of shape λ

$$s_\lambda = \sum_{T \in SSYT_\lambda} x^T$$

Schur functions and representation theory

Schur functions correspond to the irreducible representations of the symmetric group.

A nonnegative integer linear combinations of Schur functions (a.k.a. **Schur positive** function) correspond to representations of the symmetric group.

Fundamental quasisymmetric functions

For a set $S = \{s_1 < s_2 < \cdots < s_{k-1}\} \subseteq [n-1]$,

$$F_{n,S} = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ i_j < i_{j+1} \forall j \in S}} x_{i_1} \cdots x_{i_n}$$

$$F_{4,\{1\}} = x_1x_2x_2x_2 + x_1x_2x_2x_3 + x_1x_2x_3x_3 + x_1x_2x_3x_4 + \cdots$$

$$F_{4,\{2\}} = x_1x_1x_2x_2 + x_1x_2x_2x_2 + x_1x_1x_2x_3 + x_1x_2x_3x_4 + \cdots$$

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Proposition (Gessel)

The quasisymmetric function

$$\mathcal{Q}_{\mathfrak{S}_n}(\mathbf{x}) = \sum_{\sigma \in \mathfrak{S}_n} F_{n,\mathrm{DES}(\sigma)}$$

is symmetric and moreover Schur positive.

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The quasisymmetric function

$$\mathcal{Q}_{\mathfrak{S}_n}(\mathbf{x}) = \sum_{\sigma \in \mathfrak{S}_n} F_{n,\mathrm{DES}(\sigma)} = \sum_{\lambda \vdash n} |SYT_\lambda| s_\lambda = (x_1 + x_2 + \cdots)^n$$

is symmetric and moreover Schur positive. $\mathcal{Q}_{\mathfrak{S}_n}(\mathbf{x})$ corresponds to the action of left multiplication of $\mathfrak{S}_n$ on itself.

Schur positive (classical) descent enumerators

For a subset $A \subseteq \mathfrak{S}_n$, define

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What subsets of permutations produce symmetric functions?
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- Chow, Grinberg–Stanley: All permutations

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Corollary (C. +26)

Let Π be a finite set of patterns such that $\mathrm{SDES}(\pi) = \emptyset$ for all $\pi \in \Pi$. Then, $\mathcal{SQ}_{\mathrm{Av}_n(\Pi)}(\mathbf{x})$ is symmetric and Schur positive.

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This is implied by the theorem stated earlier.

Example

Let $A = \{2341, 2413, 3142, 3421, 4123, 4312\}$
(all permutations of cycle type (4))

$$\mathcal{Q}_A = s_{2,1,1} + s_{3,1} \qquad \mathcal{SQ}_A = F_{4,\{1\}} + F_{4,\{3\}} + 4F_{4,\{\emptyset\}}$$

Let $B = \{123, 213, 132, 312, 321\} = \mathrm{Av}_3(231)$

$$\mathcal{Q}_B = F_{3,\{1,2\}} + 2F_{3,\{2\}} + F_{3,\{1\}} + F_{3,\emptyset} \qquad \mathcal{SQ}_B = s_{1,1,1} + s_{2,1} + 2s_3$$

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Conclusion: $\mathcal{Q}_A$ and $\mathcal{SQ}_A$ are very different

Proof

Using the following symmetric function identity of Gessel

$$s_{(n-i,1^i)} = \sum_{\substack{S\subseteq[n-1]\\|S|=i}} F_{n,S},$$

if $a_k^{\Pi} = |\{\sigma \in \mathrm{Av}_k(\Pi) : \mathrm{SDES}(\sigma) = \emptyset\}|$, then we see

$$\begin{aligned}
\sum_{\sigma\in\mathrm{Av}_n(\Pi)} F_{n,\mathrm{SDES}(\sigma)} &= \sum_{S\subseteq[n-1]} F_{n,S}|\{\sigma \in \mathrm{Av}_n(\Pi) : \mathrm{SDES}(\sigma) = S\} \\
&= \sum_{S\subseteq[n-1]} a_{n-|S|}^{\Pi} F_{n,S} \\
&= \sum_{i=0}^{n-1} a_{n-i}^{\Pi} \sum_{\substack{S\subseteq[n-1]\\|S|=i}} F_{n,S} \\
&= \sum_{i=0}^{n-1} a_{n-i}^{\Pi} s_{(n-i,1^i)}.
\end{aligned}$$

□

Refined expansion

Let $\sigma \in \mathfrak{S}_n$ be a permutation and set $\mathrm{des}(\sigma) = |\mathrm{DES}(\sigma)|$ and $\mathrm{sdes}(\sigma) = |\mathrm{SDES}(\sigma)|$. Also set $\mathrm{asc}(\sigma) = n - 1 - \mathrm{des}(\sigma)$ i.e. the number of **ascents** of σ.

Corollary (C. +26)

If Π is a set of patterns such that $\mathrm{sdes}(\pi) = 0$ for all $\pi \in \Pi$, then for all $n \geq 1$, we have

$$\sum_{\sigma \in \mathrm{Av}_n(\Pi)} q^{\mathrm{des}(\sigma) - \mathrm{sdes}(\sigma)} t^{\mathrm{asc}(\sigma)} F_{n,\mathrm{SDES}(\sigma)} = \sum_{i=1}^{n} a_i^{\Pi}(q,t) s_{i,1^{n-i}}$$

where

$$a_i^{\Pi}(q,t) = \sum_{\substack{\sigma \in \mathrm{Av}_i(\Pi) \\ \mathrm{sdes}(\sigma) = 0}} q^{\mathrm{des}(\sigma)} t^{\mathrm{asc}(\sigma)}.$$

Refined Expansion

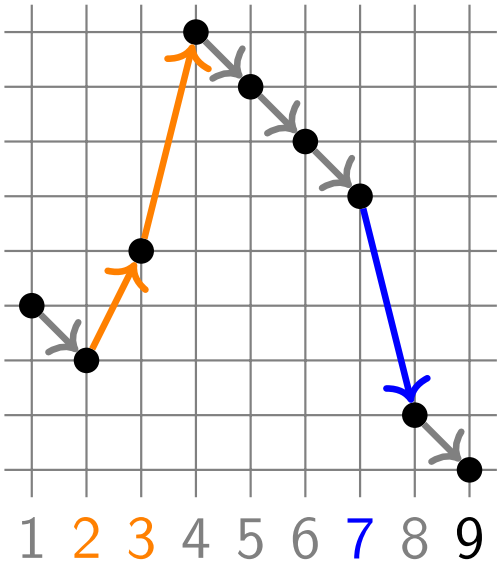
The coefficients $a_i^{\Pi}(q,t)$ of $s_{i,1^{n-i}}$ do not depend on n.

Example ($\Pi = \{231\}$)

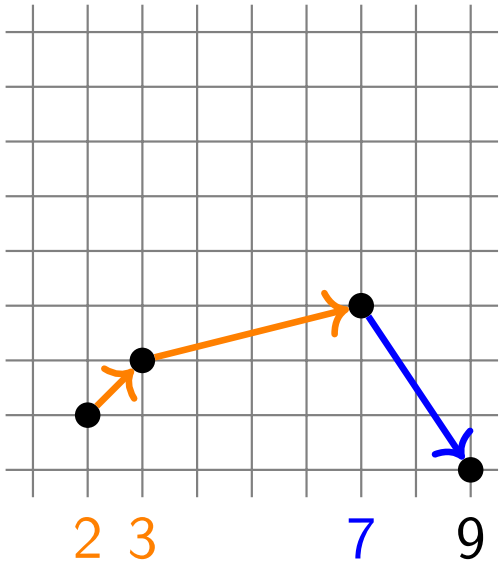
1. s_1
2. $s_{1,1} + ts_2$
3. $s_{1,1,1} + ts_{2,1} + \left(qt + t^2\right) s_3$
4. $s_{1,1,1,1} + ts_{2,1,1} + \left(qt + t^2\right) s_{3,1} + \left(3qt^2 + t^3\right) s_4$
5. $s_{1,1,1,1,1} + ts_{2,1,1,1} + \left(qt + t^2\right) s_{3,1,1} + \left(3qt^2 + t^3\right) s_{4,1} + \left(2q^2t^2 + 6qt^3 + t^4\right) s_5$
6. $s_{1,1,1,1,1,1} + ts_{2,1,1,1,1} + \left(qt + t^2\right) s_{3,1,1,1} + \left(3qt^2 + t^3\right) s_{4,1,1} + \left(2q^2t^2 + 6qt^3 + t^4\right) s_{5,1} + \left(10q^2t^3 + 10qt^4 + t^5\right) s_6$
7. $s_{1,1,1,1,1,1,1} + ts_{2,1,1,1,1,1} + \left(qt + t^2\right) s_{3,1,1,1,1} + \left(3qt^2 + t^3\right) s_{4,1,1,1} + \left(2q^2t^2 + 6qt^3 + t^4\right) s_{5,1,1} + \left(10q^2t^3 + 10qt^4 + t^5\right) s_{6,1} + \left(5q^3t^3 + 30q^2t^4 + 15qt^5 + t^6\right) s_7$

Refined expansion–Proof

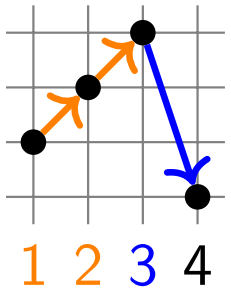
The earlier bijection preserves big descents and ascents



(a) σ



(b) u (spread out)



(c) u

Coefficient polynomials

What are the coefficient polynomials?

$$a_i^{\Pi}(q,t) = \sum_{\substack{\sigma \in \mathsf{Av}_i(\Pi) \\ \mathsf{sdes}(\sigma)=0}} q^{\mathsf{des}(\sigma)} t^{\mathsf{asc}(\sigma)}.$$

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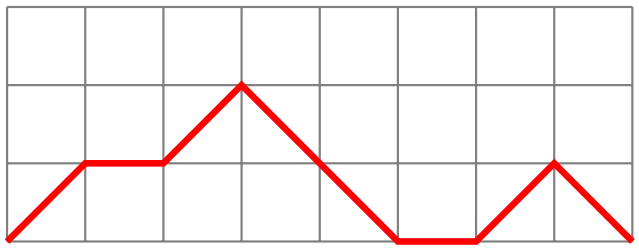

Motzkin path $P \in M_8$

Theorem

Let $\#D(P)$ and $\#F(P)$ be set of down steps and flat steps of a Motzkin path P. Then for all $i \geq 1$,

$$a_i^{231}(q,t) = \sum_{\substack{\sigma \in \mathrm{Av}_i(\Pi) \\ \mathrm{sdes}(\sigma)=0}} q^{\mathrm{des}(\sigma)} t^{\mathrm{asc}(\sigma)} = \sum_{P \in M_{i-1}} q^{\#D(P)} t^{\#F(P)+\#D(P)}$$

Switching gears

For a finite subset $A \subseteq \mathbb{Z}^n_{>0}$, define

$$\mathcal{Q}_A(\mathbf{x}) = \sum_{\sigma \in A} F_{n,\mathrm{DES}(\sigma)}$$

What set of words produce symmetric functions? Which are Schur positive?

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We will see that $\mathcal{SQ}_A(\mathbf{x})$ is Schur positive for the set of *parking functions*.

Parking functions

A **parking function** of length n is a word $w = w_1 \cdots w_n$ (i.e. a function $w : [n] \to \mathbb{Z}_{>0}$) such that $|w^{-1}([i])| \geq i$ for all i. Let PF_n denote the set of parking functions of length n.

$$\mathrm{PF}_3 = \{111, 112, 121, 211, 113, 131, 311, 122, 212, 221,$$
$$123, 132, 213, 231, 312, 321\}$$

Parking Functions

- Let $\mathrm{sasc}(w)$ be the number of $i \in [n-1]$ such that $w_i = w_{i+1} - 1$ (**small ascents**)
- Let $\mathrm{tie}(w)$ be the number of $i \in [n-1]$ such that $w_i = w_{i+1}$ (**ties**)

Theorem

For all $n \geq 1$,

$$\sum_{w \in \mathrm{PF}_n} q^{\mathrm{sasc}(w)} t^{\mathrm{tie}(w)} F_{n,\mathrm{SDES}(w)} = \sum_{i=1}^{n} (q+t+n-2)^{i-1} s_{i,1^{n-i}}$$

Proof involves passing to "Prüfer codes."

Parking Functions

The Prüfer code proof actually reveals a surprising symmetry

Theorem

For all $n \geq 1$, any permutation of the statistics defines the same, Schur positive function

$$\sum_{i=1}^{n}(q+t+n-2)^{i-1}s_{i,1^{n-i}} = \begin{cases} \sum_{w\in \mathrm{PF}_n} q^{\mathrm{sasc}(w)}t^{\mathrm{tie}(w)}F_{n,\mathrm{SDES}(w)} \\ \sum_{w\in \mathrm{PF}_n} q^{\mathrm{tie}(w)}t^{\mathrm{sasc}(w)}F_{n,\mathrm{SDES}(w)} \\ \sum_{w\in \mathrm{PF}_n} q^{\mathrm{sdes}(w)}t^{\mathrm{sasc}(w)}F_{n,\mathrm{TIE}(w)} \\ \vdots \end{cases}$$

Let $\mathcal{SQ}_{\mathrm{PF}_n}(q,t)$ denote this common value.

Expansion in the p (for parking function) basis

For a partition $\lambda = (1^{m_1} 2^{m_2} \cdots)$, let $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$

Theorem

Let y be an indeterminate. Then for all $n \geq 1$,

$$\sum_{i=1}^{n} y^{i-1} s_{i,1^{n-i}} = \frac{1}{y+1} \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} \prod_{i=1}^{\ell(\lambda)} (y^{\lambda_i} - (-1)^{\lambda_i}).$$

Applies the Adin–Roichman s-to-p formula to hook shapes

Corollary

For all $n \geq 1$,

$$\mathcal{SQ}_{\mathrm{PF}_n}(q,t) = \frac{1}{q+t+n-1} \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} \prod_{i=1}^{\ell(\lambda)} ((q+t+n-2)^{\lambda_i} - (-1)^{\lambda_i})$$

Future Directions

- Our result

$$\sum_{\sigma \in \mathrm{Av}_n(\Pi)} q^{\mathrm{des}(\sigma)-\mathrm{sdes}(\sigma)} t^{\mathrm{asc}(\sigma)} F_{n,\mathrm{SDES}(\sigma)} = \sum_{i=1}^{n} a_i^{\Pi}(q,t) s_{i,1^{n-i}}$$

says that there is an action of $\mathfrak{S}_n$ on $\mathrm{Av}_n(\Pi)$ that preserves $(\mathrm{des}(\sigma) - \mathrm{sdes}(\sigma), \mathrm{asc}(\sigma))$. It would be very interesting to actually construct it.

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Thanks!