

Almost Distant Monotone Patterns

Permutation Patterns 2026

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Definitions

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- Let $q = q_1 \dots q_k \in \mathfrak{S}_k$, we call $q' = q_1 \dots q_{j-1} \square q_j \dots q_k$ a **distant pattern*** of length k . An occurrence of a q' in p is a pattern of $p_{i_1}, \dots, p_{i_k}$ where $i_j \neq i_{j-1} + 1$, i.e. the $(j-1)^{\text{st}}$ and j^{th} entries are not adjacent.
- Example: 54**123** avoids $1 \square 23$, but **2134** does not.

Connection to Classical Patterns

Theorem (Dimitrov, 2023)

The avoidance of $q = q_1 q_2 \dots q_{i-1} \square q_i \dots q_k$ is equivalent to the simultaneous avoidance of $k + 1$ classical patterns of length $k + 1$.

Example: $1 \square 234 = \{\textcolor{red}{2}\textcolor{blue}{1}345, \textcolor{red}{1}2\textcolor{blue}{3}45, \textcolor{red}{1}3\textcolor{blue}{2}45, \textcolor{red}{1}4\textcolor{blue}{2}35, \textcolor{red}{1}5\textcolor{blue}{2}34\}$

Important Definition

We define an **almost distant pattern** of length k $q_1 q_2 \dots q_{i-1} \boxed{j} q_i \dots q_k$ is the collection of classical patterns equivalent to $q_1 q_2 \dots q_{i-1} \square q_i \dots q_k$ except the pattern p where $p_i = j$.

Example: $1 \boxed{3} 234 = \{21345, 12345, \cancel{13245}, 14235, 15234\}$

Known Results

In 2023, Bóna and Pantone studied almost distant monotone patterns where the gap condition follows the pattern (i.e. $i = k + 1$), and proved the following results:

Theorem (Bóna and Pantone, 2023)

For all $k \geq 2$ and $2 \leq j \leq k$, $L(12 \dots k \boxed{j}) = (k - 1)^2$, and $L(12 \dots k \boxed{k + 1}) = (k - 1)^2 + 1$.

Example: $L(1234 \boxed{3}) = 9$ and $L(1234 \boxed{5}) = 10$.

First Result

Theorem 1 (V, 2026+)

Let $k \geq 1$, then for all $1 \leq j \leq (k + 1)$, the almost distant patterns $12 \dots (j - 1) \boxed{j} j \dots k$ are Wilf-Equivalent.

Example: For $k = 4$, the theorem says $\text{Av}_n(\boxed{1}1234) = \text{Av}_n(1\boxed{2}234) = \text{Av}_n(12\boxed{3}34) = \text{Av}_n(123\boxed{4}4) = \text{Av}_n(1234\boxed{5})$ for all $n \geq 0$.

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Let $k \geq 1$. Then for any monotone distant pattern* q of length k , $L(q) \in [(k - 1)^2, (k - 1)^2 + 1]$.

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Theorem 1 turns out to be equivalent to a result from Bloom and Sagan (2020), relating to partial shuffles.

Another Wilf-Equivalence

We can also show that this set is equivalent to another almost distant pattern:

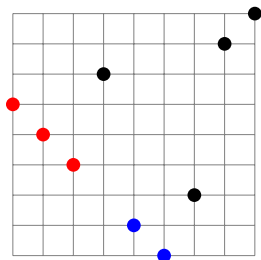
Corollary (V, 2026+)

Let $k \geq 2$. Then $1\boxed{1}2\dots k$ and $1\boxed{2}2\dots k$ are Wilf-Equivalent.

We prove this by a simple bijection, using the required structure of the entries whose values are restricted.

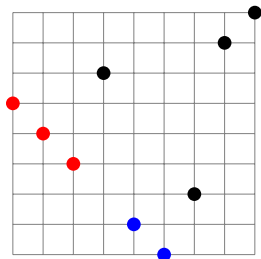
Proof Sketch

Let p avoid $1\boxed{1}2\dots k$. Consider the first two elements of a $12\dots k$ pattern, p_i, p_j . Any elements between them must be decreasing.

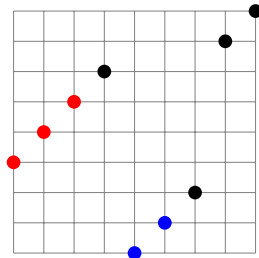


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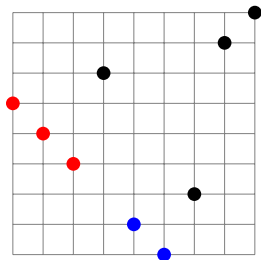


Similarly, if q avoids $1\boxed{2}2\dots k$, and q_i, q_j are the first two elements of a $12\dots k$ pattern, any elements between them must be increasing.

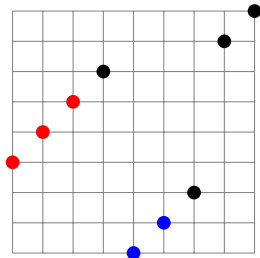


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The map simply reverses those factors. ■

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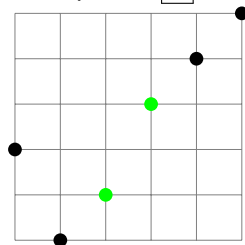
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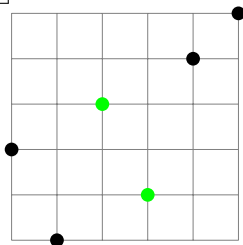
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Example: $12\boxed{3}34 \rightarrow 12\boxed{2}34$



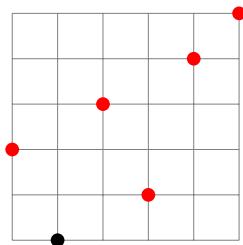
312456

$\mapsto$



314256

$=$

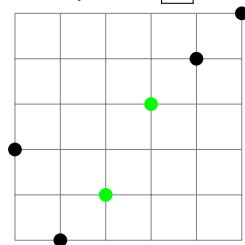


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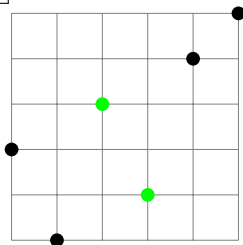
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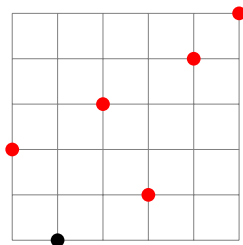
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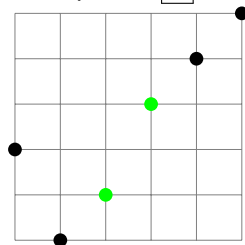
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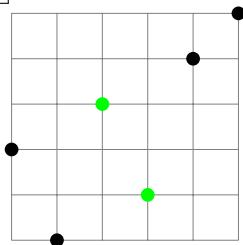
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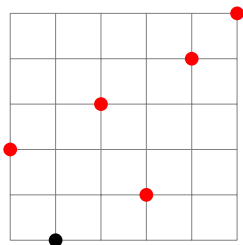
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So can we find a solution? Turns out, yes.

Second Theorem

Theorem 2 (V, 2026+)

Let $k \geq 2$, $2 \leq i \leq (k + 1)$. There exists a finite set of permutations S_i such that $\text{Av}_n(12 \dots (i - 1) \boxed{i - 1} i \dots k) = \text{Av}_n(12 \dots (i - 1) \boxed{i} i \dots k, (S_i \oplus I_{k+2-i}))$ for all $n \geq 0$. (Where I_t is the increasing pattern $12 \dots t$.)

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The proof is by double inclusion, we define what the set S_i looks like and show that the images of both sets must lie inside the other set.

Examples

Corollary (V, 2026+)

Let $k \geq 2$. Then $\text{Av}_n(12\boxed{2}3\dots k) = \text{Av}_n(12\boxed{3}3\dots k, 312 \oplus I_{k-1})$ for all $n \geq 0$.

Corollary (V, 2026+)

Let $k \geq 3$. Then

$\text{Av}_n(123\boxed{3}4\dots k) = \text{Av}_n(123\boxed{4}4\dots k, ((1423, 45123, 35124) \oplus I_{k-2}))$
for all $n \geq 0$.

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Of particular note is that for all $i > 2$, the resultant set $S_i \oplus I_{k+2-i}$ is non-empty. Therefore, these are all finite unbalanced Wilf-equivalences.

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- Growth rates? We know $L(1 \boxed{1} 2 \dots k) = (k - 1)^2 + 1$, and $L(1 \dots k \boxed{k}) = (k - 1)^2$, how does it change in-between?

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- What results can we get for values other than $i, i \pm 1$? Can we get more unbalanced Wilf-equivalences? Or even nicer results?

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- What about non-monotone almost distant patterns? Bona proved $213\dots k\boxed{k+1}$ is Wilf-equivalent to $123\dots k\boxed{k+1}$, can that be extended?

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- What about non-monotone almost distant patterns? Bona proved $213\dots k\boxed{k+1}$ is Wilf-equivalent to $123\dots k\boxed{k+1}$, can that be extended?
- Examination of all almost distant patterns of length 4 suggests more Wilf-equivalences, some that seem explainable by the method used for Theorem 1, but others that don't.

End



Thanks to the NSF for funding, and thank you for your time.

Any Questions?

Description of S_i

We can describe S_i as the minimal set of permutations p such that (a) p contains a pattern in $12 \dots (i-2) \boxed{i-1}$, and (b) the last entry in that pattern is the only element in p of rank $(i-1)$.

References

- S. Dimitrov, On permutation patterns with constrained gap sizes. *Australasian J. Combinatorics* 87(1) (2023), 98-128
- M. Bóna, J. Pantone. Permutations avoiding sets of patterns with long monotone subsequences. *J. Symbolic Comput.* 116 (2023), 130-138
- J. S. Bloom, B. E. Sagan. Revisiting pattern avoidance and quasisymmetric functions. *Ann. Comb.* 24(2) (2020), 337–361