

Statistics and Pattern Avoidance on Desarrangements

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Joint work with the 2024 DREAM Team



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DREAM

Discovering Research and Expanding Access to Mathematics

Permutations and desarrangements

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Let $\mathfrak{D}_n$ be the set of desarrangements in $\mathfrak{S}_n$.

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- 3 Remove the parentheses.

Example (cont.): $\Phi(\pi) = 756943182$

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Summary of our work:

① Refined enumeration!

We counted permutations in $\mathfrak{D}_n$ with respect to various permutation statistics.

② Pattern avoidance!

We counted permutations in $\mathfrak{D}_n(\Pi)$ for every $\Pi \subseteq \mathfrak{S}_3$.

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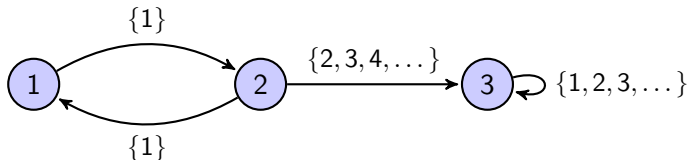
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The descent compositions of (nondecreasing) desarrangements are precisely those obtained by traversing this digraph from 1 to 3:



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graphs encoding descent compositions



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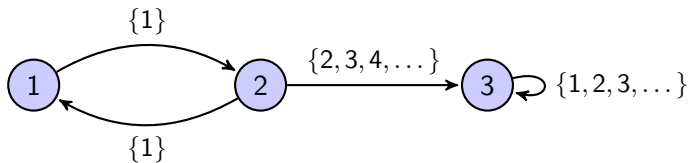
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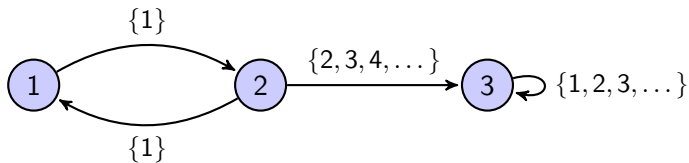
Let’s just walk through a couple examples involving desarrangements.

Warm-up: e.g.f. for derangements



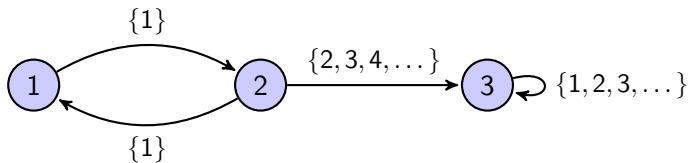
$$\text{Let } A = \left(I_3 + \begin{bmatrix} 0 & x & 0 \\ x & 0 & \frac{x^2}{1-x} \\ 0 & 0 & \frac{x}{1-x} \end{bmatrix} \right)^{-1}$$

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Let $\hat{A}$ be the corresponding matrix of exponential generating functions:

$$\hat{A} = \begin{bmatrix} \cosh x & -\sinh x & \sinh x - x \\ -\sinh x & \cosh x & 1 - \cosh x \\ 0 & 0 & 1 - x \end{bmatrix}.$$

Warm-up: e.g.f. for desarrangements (cont.)

Take the inverse again:

$$\hat{A}^{-1} = \begin{bmatrix} \cosh x & \sinh x & \frac{x \cosh x - \sinh x}{1-x} \\ \sinh x & \cosh x & \frac{1 - \cosh x + x \sinh x}{1-x} \\ 0 & 0 & \frac{1}{1-x} \end{bmatrix}$$

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Adding $\cosh x$ (to account for the decreasing desarrangements) and simplifying yields

$$\sum_{n=0}^{\infty} |\mathcal{D}_n| x^n = \frac{e^{-x}}{1-x}.$$

Counting desarrangements by double ascents

We call $2 \leq k \leq n-1$ a **double ascent** of $\pi \in \mathfrak{S}_n$ if $\pi_{k-1} < \pi_k < \pi_{k+1}$.

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Fact

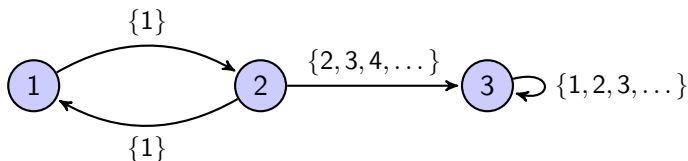
Each increasing run of length $k \geq 2$ contributes $k - 2$ double ascents.

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Fact

Each increasing run of length $k \geq 2$ contributes $k-2$ double ascents.



So each part of size $k \geq 2$ contributes a weight of t^{k-2} :

$$A = \left(I_3 + \begin{bmatrix} 0 & x & 0 \\ x & 0 & \frac{x^2}{1-tx} \\ 0 & 0 & x + \frac{x^2}{1-tx} \end{bmatrix} \right)^{-1}.$$

Counting desarrangements by double ascents (cont.)

Applying the run theorem in the same way as before yields:

Theorem (Bsila–Cox–Hugo–Styron–Z. 2025)

We have

$$\sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{D}_n} t^{\text{dasc}(\pi)} \frac{x^n}{n!} \\ = \frac{e^{-x} \left((2-t)\alpha \cosh\left(\frac{\alpha x}{2}\right) - t(1-t) \sinh\left(\frac{\alpha x}{2}\right) \right) + (1-t)\alpha e^{(1-t)x/2}}{(3-2t) \left(\alpha \cosh\left(\frac{\alpha x}{2}\right) - (1+t) \sinh\left(\frac{\alpha x}{2}\right) \right)}$$

where $\alpha = \sqrt{(t+3)(t-1)}$.

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where $\alpha = \sqrt{(t+3)(t-1)}$.

We have analogous results for the descent number, peak number, valley number, double descent number, and a few joint distributions.

Desarrangements avoiding a single length 3 pattern

All pattern avoidance results were proven using routine combinatorial and generating function methods.

Theorem (Bsila–Cox–Hugo–Styron–Z. 2025)

Π	$ \mathfrak{D}_n(\Pi) $	OEIS
$\{321\}$	C_{n-1} (Catalan numbers)	A000108
$\{132\}$	F_{n+1} (Fine numbers)	A000957
$\{231\}$		
$\{123\}$		
$\{213\}$	$\sum_{k=1}^{n-1} (-1)^{n-k-1} C_k$	A033297
$\{312\}$		

Desarrangements avoiding two length 3 patterns

Theorem (Bsila–Cox–Hugo–Styron–Z. 2025)

Π	$ \mathfrak{D}_n(\Pi) $	OEIS
$\{132, 321\}$	$n - 1$	A000027
$\{132, 231\}$	2^{n-2}	A000079
$\{231, 321\}$		
$\{312, 321\}$	2^{n-3}	
$\{123, 213\}$	J_{n-1} (Jacobsthal numbers)	A001045
$\{132, 213\}$		
$\{213, 231\}$		
$\{132, 312\}$		
$\{231, 312\}$		
$\{123, 132\}$	$\frac{2^{n+1} + (-1)^n(7 - 3n)}{9}$	A113954
$\{123, 231\}$	$\frac{2n(n - 1) + (-1)^n(5 - 2n) + 3}{8}$	A130404
$\{123, 312\}$	$\left\lceil \frac{(n - 1)^2}{4} \right\rceil$	A004652

Desarrangements avoiding three length 3 patterns

Theorem (Bsila–Cox–Hugo–Styron–Z. 2025)

Π	$ \mathfrak{D}_n(\Pi) $	OEIS
$\{132, 231, 321\}$	$n - 1$	A000027
$\{123, 132, 231\}$	$\begin{cases} n - 1, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$	A124625
$\{123, 132, 312\}$	$\left\lfloor \frac{n}{2} \right\rfloor$	A008619
$\{123, 231, 312\}$		
$\{123, 213, 231\}$		
$\{132, 213, 231\}$		
$\{132, 231, 312\}$		
$\{123, 132, 213\}$	f_{n-1} (Fibonacci numbers)	A000045
$\{231, 312, 321\}$		

(For any $\Pi \subseteq \mathfrak{S}_3$ that did not appear in these tables, the enumeration of $\mathfrak{D}_n(\Pi)$ is trivial.)

Back to derangements

Robertson–Saracino–Zeilberger (2002) and Mansour–Robertson (2002) studied the distribution of fixed points over $\mathfrak{S}_n(\Pi)$ with $\Pi \subseteq \mathfrak{S}_3$ and $1 \leq |\Pi| \leq 3$.

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Let $\mathcal{D}_n(\Pi)$ be the set of derangements in $\mathfrak{S}_n(\Pi)$.

Theorem (Bsila–Cox–Hugo–Styron–Z. 2025)

Let $\Pi \subseteq \mathfrak{S}_3$ with $1 \leq |\Pi| \leq 3$. Then

$$|\mathcal{D}_n(\Pi)| = |\mathfrak{D}_n(\Pi)| \quad \text{for all } n \geq 0$$

if and only if Π is one of the following: $\{132\}$, $\{132, 312\}$, $\{132, 321\}$, $\{213, 231\}$, $\{123, 132, 312\}$, $\{123, 213, 231\}$, $\{123, 312, 321\}$, $\{132, 312, 321\}$, $\{213, 231, 312\}$, or $\{213, 231, 321\}$.

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A bijective proof was provided by Henke–Stephens–Z. (2026+).

A related conjecture

Every permutation π can be uniquely decomposed as $\pi = \iota\delta$, where ι is an increasing subsequence and δ is a desarrangement.

Example

Given $\pi = 46785213$, we have $\iota = 467$ and $\delta = 85213$.

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Let $\Pi \subseteq \mathfrak{S}_3$ with $1 \leq |\Pi| \leq 3$. Then

fix and pix are equidistributed over $\mathfrak{S}_n(\Pi)$ for all $n \geq 0$

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Proved independently by Dong–Xu (2026+) and Henke–Stephens–Z. (2026+).

Avoiding a length 3 pattern and a length 4 pattern



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For her 2025 honors thesis, Caroline studied the enumeration of $\mathfrak{D}_n(\sigma, \tau)$ where $\sigma \in \mathfrak{S}_3$ and $\tau \in \mathfrak{S}_4$.

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Out of the 82 nontrivial possibilities for σ and τ , Caroline enumerated $\mathfrak{D}_n(\sigma, \tau)$ for 50 of them.

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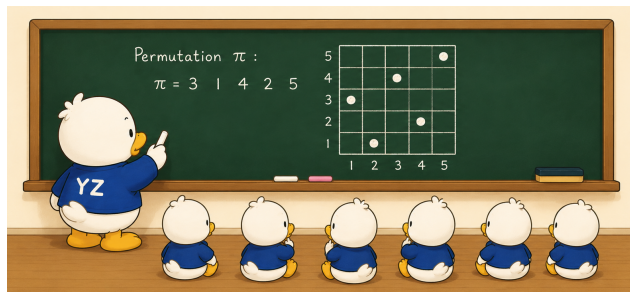
Sequences involving Fibonacci numbers feature in 27 of these.

The last slide

Chadi Bsila, Caroline E. Cox, Anna S. Hugo, Lindsey A. Styron, and Yan Zhuang.

Desarrangements revisited: statistics and pattern avoidance.

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THANK YOU!