



LOS ANGELES
JULY 20-24, 2026

Permutation Patterns 2026
California State University, Los Angeles
July 20–24, 2026

Conference sponsors

Permutation Patterns 2026 is grateful for the generous support from the United States National Science Foundation (NSF) and The Combinatorics Consortium (TCC).



Committees

- **Organizing committee:** Jason O'Neill, Justin Troyka
- **Program committee:** Frédérique Bassino, Michal Opler, Jay Pantone, Justin Troyka
- **Travel support committee:** Pamela Harris, Jay Pantone, Bridget Tenner, Vince Vatter
- **Permanent steering committee:** Michael Albert, Mathilde Bouvel, Alexander Burstein, Luca Ferrari, Jay Pantone, Lara Pudwell

Conference talks and posters

- All conference talks and the poster session will take place in Rosser 132. This is a room connected to Rosser Hall which has a door that opens to the outside.
- On most days of the conference, the talks will run from 9:00am to 5:00pm. We are ending early on Wednesday and on Friday.
- There will be an open problem session on Tuesday afternoon and a poster session on Thursday afternoon.

Dining options on campus

The Village Commons is a buffet-style (all you can eat) dining hall connected to the South Village residence hall. Breakfast is \$16.00, lunch is \$18.00, and dinner is \$19.25. Vegan/vegetarian options are available. Pay by credit/debit card upon entry to the dining hall for each meal.

There are various other dining options on our campus. Here are their summer hours:

- **Diablo Grill:** M-Th, 8 a.m.-1:30 p.m. and F, 8 a.m.-1 p.m.
 - Order online <https://diablo-express.square.site/> and pick up in the food court
- **Carl's Jr** (food court, attached to the Golden Eagle building): M-F, 8 a.m.-2 p.m.
- **Everytable** (in the library): M-F, 9 a.m.-3 p.m.
- **Juice It Up** (in the Golden Eagle building): M-F, 10 a.m.-3 p.m.
- **Sbarro** (in the Student Union): M-F, 8 a.m.-2 p.m.
- **Starbucks** (in the Student Union): M-Th, 7 a.m.-1 p.m. and F, 7 a.m.-12 p.m.
- **Bookstore:** M-F, 8:30 a.m.-2 p.m.

Conference excursion

The conference excursion will be after lunch on Wednesday, July 22. After the talks end at 12:00pm, you will be given a boxed lunch, and we will leave campus at about 12:30pm. You have two options:

- (a) **Descanso Gardens**, a botanical garden in La Cañada Flintridge. We have a coach bus which will take us there and back, a drive of 30 minutes each way. If you opt for this, we will give you a complimentary ticket. “At Descanso, our gardens and collections represent both the diversity of Southern California plants and an examination of the history of our local landscape. As you stroll through a mix of wild and cultivated land, you’ll find California native plants, an evolving oak woodland, and one of the largest camellia collections in the United States.”
- (b) **Exposition Park**, an urban park with free museums, right next to Downtown LA. The California Science Center and the California African American Museum are both free of charge. You may also choose to purchase admission to the Natural History Museum for \$18 (or \$14 for students). Also nearby is the beautiful campus of University of Southern California, as well as a number of shops and restaurants. We are not arranging transportation to Exposition Park, but the Metro J Line (Silver Line) bus will get you there in 30 minutes for \$1.75 each way.

Please inform conference organizers of your choice.

Conference banquet

The banquet will happen on Thursday, July 23rd in the evening. We will stay true to our Los Angeles roots and have a Taco Bar for the banquet! Participants will have a drink ticket for wine/beer as well as Aqua Fresca. Appetizers will start being served at 5:30pm with dinner served at 6:30pm.

- **Appetizers:** Caesar Salad, Shrimp Ceviche, Chicken Taquitos, Mini Spicy Elote Empandas, and Vegan Samosa.
- **Main Course:** Chicken, Carne Asada, Tofu/Fajita Mix (for those who are Vegan/Vegetarian) to put in corn tortillas with Rice, Refried Beans, Onion, Cilantro, Lime Wedges, and Salsa.
- **Dessert:** Cookies with Vegan and GF options for those who disclosed dietary restrictions.

Anti-harassment policy

The organizers of Permutation Patterns 2026 seek to create a safe and welcoming environment for people of all genders, sexual orientations, gender identities, races, ethnicities, and nationalities. We will not tolerate discrimination or harassment on the basis of those categories or on any other basis.

If you experience or witness conduct not in keeping with the spirit of this statement, please contact a conference organizer, and we will do everything we can to remedy the situation.



KEY

- 1 Theatre (State Playhouse)
- 1A Music Hall
- 2 Music (Arena Theatre)
- 3 King Hall
- 4 University-Student Union Plaza
- 5 University-Student Union
- 6 Golden Eagle
- 7 University Library
- 8 Recreational Fields
- 9 Fine Arts
- 9A Ronald H. Silverman Fine Arts Gallery
- 10 Physical Education (Gymnasium)
- 11 Engineering & Technology
- 11A Advanced Materials and Manufacturing Laboratory
- 12 Student Services Building
- 13 Biological Sciences
- 14 Student Health Center
- 15 Simpson Tower
- 15A Salazar Hall
- 15B Simpson Annex
- 17 Career Center / Center for Engagement, Service, and the Public Good
- 18 Stadium (Jesse Owens Track)
- 19 Los Angeles Football Club Performance Center
- 20 LACHSA
- 23 Corporation Yard
- 24 Reeder Field
- 24A Tennis / Basketball Courts
- 26 Stern MASS High School
- 27A La Kretz Hall
- 27B James M. Rosser Hall
- 29 Luckman Theatre
- 29A Luckman Fine Arts Gallery
- 29B Intimate Theatre
- 30 Anna Bing Arnold Children's Center
- 34 Housing Phase I
- 36 Housing Phase II
- 37 Golden Eagle Apartments
- 43 TV, Film & Media Center
- 43 Hertzberg-Davis Forensic Science Center
- 46 Public Safety / Parking Services
- 47 WELCOME CENTER
- 48 Hydrogen Station
- 49 Rongxiang Xu Bioscience Innovation Center
- 50 South Village (Residence Hall)

PARKING

- Structure
- Lot
- Transportation Service Center (Located on 2nd floor of Student Services Building)

*For a braille tactile map of Cal State LA please contact accessibility@calstatela.edu or call 323-343-2750".

Parking nearest to South Village residence (Lot 5 or Structure C or Structure E)

Parking nearest to Rosser 132 (Structure B, level C)

- A. Metro J line Busway (take the stairs/elevator)
- B. Other buses, at Cal State LA Transit Center
- C. Bridge to residence hall and dining hall
- D. South Village (residence hall) main entrance
- E. Village Commons (dining hall)
- F. Rosser 132 (where the talks are) – the door opens directly to the outside
- G. Food court
- H. University Club (where the banquet is)
- BJK. Statue of Billie Jean King
- GE. Golden Eagle statue

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— Plenary talks —

FROM CONSECUTIVE AVOIDANCE TO PRESCRIBED INDICES: A PANORAMIC VIEW OF LOCAL ORDER IN PERMUTATIONS

Mohamed Omar

York University

Classical pattern avoidance asks whether a permutation avoids a given relative order everywhere. Consecutive pattern avoidance sharpens this by requiring the entries to be adjacent: for $\tau \in \mathfrak{S}_k$, an occurrence of τ in $\pi = \pi_1 \cdots \pi_n$ starts at i when

$$\text{std}(\pi_i \pi_{i+1} \cdots \pi_{i+k-1}) = \tau.$$

Avoidance is the case in which no such start exists. This local version of pattern avoidance has grown into a major theme of the subject, through work of Elizalde and Noy on consecutive patterns and clusters and through related contributions by Kitaev, Mansour, Burstein, Vatter, Blitvić, and others in the Permutation Patterns community. One reason for its appeal is that a very small local rule can control global enumeration, generating functions, asymptotics, and growth.

This talk follows the historical development of a further refinement: rather than asking only whether a consecutive pattern occurs, ask *where* it occurs. The guiding object is the set of starting positions of a consecutive pattern τ . Prescribing this set, or prescribing where occurrences are forbidden, turns avoidance into a positional enumerative problem. In hindsight, this idea was already visible in MacMahon's descent-set enumeration. Descents are occurrences of the consecutive pattern 21, and one writes

$$d(I; n) = \#\{\pi \in \mathfrak{S}_n : \text{Des}(\pi) = I\}$$

for the set of permutations in $\mathfrak{S}_n$ with descents precisely in positions I . Peaks give a second fundamental example: a peak at i is the local condition $\pi_{i-1} < \pi_i > \pi_{i+1}$, equivalently one of the two length-three consecutive patterns 132 or 231 centered at i . For a prescribed peak set one similarly studies $p(I; n) = \#\{\pi \in \mathfrak{S}_n : P(\pi) = I\}$. Thus the passage from avoidance to prescribed indices is not a technical embellishment; it is the step that exposes the polynomial and asymptotic structure hidden inside local restrictions.

The speaker's work will be presented as part of this development. Billey, Burdzy, and Sagan proved that, for a fixed peak set I , the count $p(I; n)$ has the form $2^{n-|I|-1} p_I(n)$,

and conjectured a positivity property for the polynomial p_I : in a natural binomial-coefficient basis, all coefficients should be nonnegative. Joint work of the speaker with Díaz-Lopez, Harris, and Insko proved this by giving a direct insertion recurrence without the cancellations present in earlier formulas. Joint work with Díaz-Lopez, Harris, Insko, and Sagan developed the parallel theory of descent polynomials $d(I; n)$, returning to MacMahon's descent-set counts with new recurrences, coefficient questions, and comparisons with peaks. These results also connect to work of Davis, Nelson, Petersen, and Tenner on pinnacle sets, where one prescribes peak values rather than peak positions.

The same viewpoint then moves beyond a line. In work with Díaz-Lopez, Everham, Harris, Insko, and Marcantonio, peaks were studied on graphs: a vertex is a peak if its label is larger than the labels on all neighboring vertices. This separates two ingredients that coincide for permutations, namely the linear order of positions and the local neighborhood in which comparisons are made. More recently, the speaker introduced X -descent sets. For a relation $X \subseteq \mathbb{N}^2$, $\text{Des}_X(\pi) = \{i : (\pi_i, \pi_{i+1}) \in X\}$ and $d_X(I; n)$ counts permutations with this prescribed set. This single notation includes ordinary descents and ascents, successions and reverse successions, and many two-letter consecutive restrictions whose behavior depends on the actual labels rather than only on relative order. In favorable cases the theory reduces to descent-free counts $d_X(\emptyset; n)$, which are equivalent to Hamiltonian path counts in a naturally associated directed graph; in periodic cases transfer-matrix methods appear.

Similar analyses can be extended to the infinite setting. With Troyka, the speaker studies $d_n(I)$ and $p_n(I)$ when an infinite set $I \subseteq \mathbb{N}$ is intersected with $[n - 1]$, asking for the exponential growth rate of the corresponding descent or peak prescription. This places local pattern constraints in an asymptotic setting and shows how a choice of indices can tune the global scale of enumeration.

In short, the talk will give a panoramic account of how prescribed-index consecutive avoidance grew out of classical pattern avoidance, descent sets, peak sets, graph labelings, and growth-rate problems. It will also present open problems and curiosities for future directions.

AN ANIMATED INTRODUCTION TO PLANCHEREL MEASURE: FROM LONGEST INCREASING SUBSEQUENCES TO JEU DE TAQUIN FORESTS ON INFINITE YOUNG TABLEAUX

Dan Romik

University of California, Davis

What is the typical length of a longest increasing subsequence in a random permutation of N elements? This simple question asked by Ulam in the 1960s has led to a rich trove of beautiful combinatorial and probabilistic ideas. In this talk I will give a guided tour to many of these ideas, using an interactive visualization app that brings the mathematics to life in an entertaining fashion. A central object of interest we will focus on is the Plancherel measure, an ensemble of random infinite Young tableaux which captures information related to longest increasing subsequences asso-

ciated with prefixes of an infinite sequence of independent random numbers sampled uniformly from $(0,1)$. This measure can be analyzed using combinatorial gadgets such as the RSK algorithm and the jeu de taquin map, and ideas from the representation theory of the symmetric group. After introducing the relevant notions, I will discuss (and illustrate graphically with fun animations) recent joint work with Piotr Śniady where we study a new combinatorial structure called the jeu de taquin forest, that reveals additional information associated with Plancherel-random Young tableaux. The forest can be given an intriguing structure of a “tree of trees” that raises many additional natural open problems.

The talk will be accessible and will assume only a general background in the combinatorics of permutations and modest familiarity with probability.

— Contributed talks —

INVERSE LIMITS OF POSETS FROM PATTERN AVOIDANCE AND OTHER SET PARTITIONS

Amrita Acharyya

University of Toledo

It is known when we call a poset P , a $\mathcal{P}$ -chain permutational poset, given a subset of permutations $\mathcal{P}$ of the symmetric group S_n [2]. Here we use the same idea to study subsets of words of length n , that are not necessarily permutations, for example: when they are certain classes of r.g.f.'s (restricted growth functions) induced by set partitions in standard form over $[n] = \{1, 2, \dots, n\}$ (correlated with various pattern avoidance classes in permutations). Varying n only, and also varying n and k (the number of blocks of the set partitions) simultaneously, we can show that those posets form a projective system of trees and lattices. These lattice structures can be extended over standard type B set partitions over $\langle n \rangle = \{-1, -2, \dots, -n, 0, 1, 2, \dots, n\}$ and also in other algebraic and combinatorial scenario. We study properties of the tree and lattice structures of these projective systems and the corresponding projective limits.

Definition 1. A poset P will be called $\mathcal{P}$ - chain - word, where $\mathcal{P}$ is a subset of the set of all words of length n , if it is possible to label the covering relations of P (i.e. the edges of the Hasse diagram of P) with numbers from $1, 2, \dots, n$ in a way that along different maximal chains of P (whose length is necessarily n) the labels form different words from $\mathcal{P}$, and every word w of $\mathcal{P}$ arises in this manner. For example, it is well known that the Boolean lattice B_n is a chain word poset, where $\mathcal{P}$, the set of words is all the words (here permutations), in one line notation in S_n .

Definition 2. [6],[7] Let F be a nonempty family of finite graphs or a family of finite rooted graphs with a fixed family of epimorphisms among the structures in F . We say that F is a projective Fraisse family if

1. F is countable up to isomorphism, that is, any sub-collection of pairwise non isomorphic structures of F is countable.
2. Epimorphisms in F are closed under composition and each identity map is in F .

3. For $B, C \in F, \exists D \in F$ and epimorphisms f and g in F such that $f : D \mapsto B$ and $g : D \mapsto C$.
4. For every two epimorphisms $f : B \mapsto A$ and $g : C \mapsto A$ in F there exist epimorphisms $f_0 : D \mapsto B$ and $g_0 : D \mapsto C$ in F such that $f \circ f_0 = g \circ g_0$.

Notations: Let, R_n be the set of all r.g.f.'s of length n , $R_n(\sigma)$ be those avoiding a pattern σ . Let $R_{n,k}$ be the set of all words in R_n with largest digit k , $R_{n,k}(\sigma)$ accordingly.

For any positive integer n , there is a natural way to create a rooted n -ary tree structure (**viewed as a poset**) labeling the top edge from the root always as 1 (the first digit of an r.g.f.'s being 1). Along different maximal chains (whose length is necessarily n) of the tree, the labels form different words from R_n . On the other hand, every word w of R_n arises in this manner, displaying any R_n as a $\mathcal{P}$ -chain poset, where $\mathcal{P} = R_n$. We see a natural embedding of R_n inside any R_m , where $m > n$ as mentioned in section 6 in [2]. Backwardly, we can find a natural projection of the Hasse diagram of that poset R_m upon R_n as an epimorphism of graphs and posets, creating a projective Fraisse Family of rooted trees (and hence a poset, if we consider the elements of the posets as the vertices of the respective trees) as in the above definition 2, along with monotone epimorphisms as in definition 3.2 in [6]. **We note that for any set of patterns σ , the similar as above happens in case of $R_n(\sigma)$, varying n over the set of all positive integers.**

We can extend those trees as a lattice in a natural way tagging a vertex ϵ in the bottom of those trees and connecting ϵ with the bottom vertices of the tree by an edge respectively. We denote the corresponding lattices from the pattern avoidance classes as $R_{n,L}(\sigma)$. We note that similar things happen for $R_{n,L}$ and $R_{n,L}(\sigma)$. Examples:

1. Consider $R_{n,k}(1/23)$ following [1]: For a given fixed positive integer n , we notice that the Hasse diagram of $R_{n,k+1}(1/23)$ contains a copy of that of $R_{n,k}(1/23)$, where $1 \leq k < n$. We can define an epimorphism of graphs $\phi_{n,k+1} : R_{n,k+1}(1/23) \rightarrow R_{n,k}(1/23)$ creating a Projective Fraisse family of trees and confluent (as in definition 3.1 in [7]) bonding epimorphisms. Extending the bonding maps in the corresponding lattice structures varying both n and k , we get a Projective Fraisse family of graphs (from the Hasse diagrams) with confluent epimorphisms. In the lattice case, if we fix $k(< n)$ and make n vary upto infinity, the corresponding bonding maps are again confluent. By proposition 6.5 and definition 6.3 in [6], the corresponding projective limit, while both n, k vary, is kelley (in case of tree). When, we fix k , and we make the n going to infinity by proposition 5.7 in [6], the corresponding inverse limit is arcwise connected (in case of tree and lattice both)
2. One can create Projective Fraisse Family of trees (and lattices) from **standard type B set partitions** of $\langle n \rangle$ as in [3], [4], [5] labeling the edges of maximal chains via s.r.g.f. (signed restricted growth functions) in a similar way. If we denote the corresponding lattice by $R_{n,k,L}^B$, then we can label the edges of maximal chains of those lattices by using the s.r.g.f. as in [4], [5] (ignoring the bottom edges connecting to ' ϵ ' shown by dashed lines). If we ignore the lattice structure, omitting

the ' ϵ ' and the edges connecting to ϵ , then we see again the corresponding projective limit (where n varies upto infinity for a fixed k) is arcwise connected (definition 2.19 in [7]), Kelley and a smooth dendroid (definition 2.26 in [7]), satisfying lifting property by [7]. Some figures are given below.

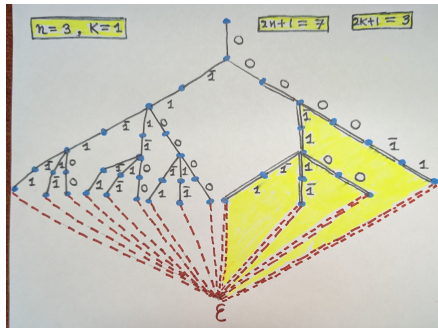


Figure 1: Lattices of type B partitions, a copy of $R_{2,1,L}^B$ is highlighted inside $R_{3,1,L}^B$, in the bottom right with double lines along the edges

3. **Lattices of non crossing partitions with 2 blocks:** We notice that if $NC(n, k)$ is the set of all non crossing partitions of $[n]$ in to k many blocks, then for $k = 2$, always $NC(n, 2)$ contains a copy of $NC(n - 1, 2)$. We see a natural epimorphism from $NC(n, 2)$ upon $NC(n - 1, 2)$, by utilising the technique of sequence of first occurrence $F(\rho)$ and it's complement $R(\rho)$ for any r.g.f. ρ as in [8]. By proposition 6.5 in [6], the corresponding projective limit N is Kelley.

Questions for future works:

1. Can we form such Projective Fraisse family of trees and lattices varying both n and k for standard type B partitions (with $1 \leq k < n$)?
2. Can we form Projective Fraisse family of lattices varying n upto infinity and fixing any k in case of non crossing partitions (and more generally varying both n and k upto infinity with $1 \leq k < n$)?

References

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- [2] Rodica Simion and Frank W Schmidt, 1985 *Restricted Permutation* published by European Journal of Combinatorics
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- [9] Amrita Acharyya, Inverse limits of various posets , arXiv:2512.12007v1

MAPPED TILINGS: A NEW APPROACH FOR AUTOMATIC ENUMERATION

Reed Acton

Reykjavik University

This talk is based on joint work with Christian Bean, Abigail Olson, Henning Ulfarsson

Introduction

Enumerating the size of permutation classes defined by pattern avoidance is a foundational goal in the field. In this talk, we expand existing methods of automatically discovering such enumerations by introducing a framework capable of enumerating complex pattern classes such as those defined by grid classes, Cayley permutations, mesh patterns, and more. We also demonstrate how the same framework can simultaneously track permutation statistics alongside the standard enumeration.

Tilings

Throughout the literature on permutation patterns, many proofs utilize an “imaginary grid” to leverage the structure of a pattern in constructing the enumeration of that pattern’s avoidance class. In [1], the authors introduced *tilings* as a formalization of that imaginary grid. In particular, a tiling is an object that defines a class of *gridded permutations* on a grid of fixed dimensions, using obstructions and requirements.

Example 1.

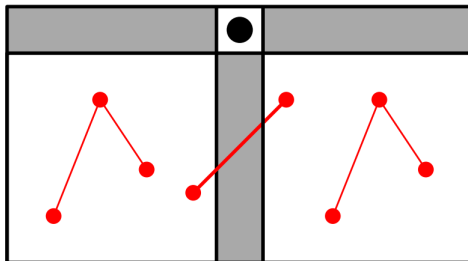


Figure 1: A 3×2 tiling T representing placing the topmost point while enumerating $\text{Av}(132)$. Obstructions (depicted in red) must be avoided by all gridded permutations in the class, and all gridded permutation in the class must contain the point in cell $(1,1)$.

Since T represents a class of gridded permutations and not standard permutations, we must denote the elements of the class defined by T with their cells. In this instance, the permutation 123 with gridding $(0,0)(0,0)(1,1)$ would live on T , but the same permutation with any other gridding would not. In cases where a there are several grillings of a permutation that live on T , each one is a distinct element of the class and counted separately in the enumeration.

Tilescope is an implementation of these tilings into the Combinatorial Exploration (CombEx) framework for automatically enumerating permutation classes defined by pattern avoidance. The output of *Tilescope* after enumerating a permutation class defined by a tiling T is a generating function whose coefficients are the number of gridded permutations of a given size that exist on T . [PermPal](#) hosts a collection several thousand of such classes enumerated by *Tilescope*, including all size-4 Wilf equivalence classes (except one). Encouraged by these successes, we set out to expand the capabilities of *Tilescope* to enumerate more complex permutation classes—such as those defined by mesh patterns or grid glasses. To accomplish this, we attach *parameters* to a base tiling to construct a new *mapped tiling* object.

Mapped Tilings

Definition 2. A *parameter* P is a tuple (G, m) where G is a c by r tiling we call the “ghost” and m is the row-column map which takes the grid cells of G and maps them to new cells. The row-column map defines a function $\phi_P : [G] \rightarrow \mathbf{G}$ that takes a gridded permutation on G and gives a new gridded permutation with the same underlying pattern.

We say a gridded permutation π *satisfies* a parameter $P = (G, m)$ if the sub-gridded permutation of π restricted to the image cells of m has a preimage under ϕ_P .

By restricting the row-column-map of a parameter $P = (G, m)$ to a chosen base tiling B , the ghost G can act on B similarly to an obstruction by excluding from the class any gridded permutations on B which satisfy P —we call such a parameter an *Avoiding Parameter*. Likewise, a *Containing Parameter* allows only those gridded permutations

on B which satisfy P -acting similarly to a requirement. For a gridded permutation π on B , we can even consider the number of “occurrences” of P by determining $|\phi_P^{-1}(\pi)|$. A parameter used to track occurrences in this way is called an *Enumerating Parameter* and would introduce an additional variable to the generating function for the enumeration of the permutation class.

Definition 3. A *mapped tiling* or *mapping* M is a tuple (B, A, C, E) where B is a tiling, and A , C , and E are sets of avoiding parameters, containing parameters, and enumerating parameters, respectively. By the above definition, a mapping M defines the class of gridded-permutations that live on B , do not satisfy any parameters in A , and satisfy every parameter in C . The generating function for $|M|$ can be defined as

$$\sum_{\pi \in M} x^{|\pi|} \cdot \prod_{E_n \in E} y_n^{|\phi_{E_n}^{-1}(\pi)|}$$

where each y_n tracks the occurrences of a specific enumerating parameter $E_n \in E$.

Using a new reconstruction of Telescope with Cayley permutations as the base object, we built *MapplePy*: a working implementation of mapped tilings into Combinatorial Exploration. So far, *MapplePy* has been able to automatically reproduce some or all results from papers such as [2], [3], and [4]. *MapplePy* can be found at <https://github.com/Ollson2921/MapplePy> and the tools for Cayley Permutation tilings can be found at <https://github.com/Ollson2921/CayleyPerms>.

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This talk is based on joint work with Aaron Geary and Christina Graves

In [5], Bóna and Smith introduce the notion of **strong pattern avoidance**, in which a permutation π and its square π^2 in the symmetric group (both written in one-line notation) both avoid a given pattern τ . In that paper they prove that the number of permutations that strongly avoid 123 is eventually zero and show that those that strongly avoid 312 (and by symmetry, 231) have a rational generating function:

$$s_{312}(x) = \frac{1 - x - x^2 + x^3}{1 - 2x - x^2 + 2x^3 - x^4}.$$

In [6], Burcroff and Defant proved that permutations that strongly avoid $123\dots k$ is eventually zero for any k . They also enumerate those permutations avoiding the sets $\{132, 3421\}$, $\{213, 4312\}$, and $\{321, 3412\}$, and in [7], Pan proves a conjecture from [6], enumerating $\{321, 1342\}$. In [3], Archer and Geary investigate those permutations for which π and π^2 have at most one descent, which is a problem that can be rephrased in terms of strong avoidance of the set $\{321, 3142, 2143\}$. Finally, in [4], Archer and Graves enumerated those permutations that strongly avoid 132, finding an algebraic generating function in terms of $c(x)$, the Catalan generating function. Enumerating those permutations that strongly avoid 321 remains open, though a lower bound is given in [5].

In [2], Archer and Geary consider a generalization of this notation called **chain pattern avoidance**, in which a permutation π is said to avoid the chain $(\tau_1 : \tau_2 : \dots : \tau_k)$ if the i -th power of π (i.e., π^i) avoids τ_i . For example, the permutation $\pi = 234567891$ avoids the chain $(312 : 132)$ since π avoids 312 and $\pi^2 = 345678912$ avoids 132. As another example, strong avoidance of the pattern 132 would correspond to avoiding the chain $(132 : 132)$.

For a more complicated example, note that $\pi = 24687531$ avoids the chain

$$(312, 213 : \emptyset : 213 : \emptyset : 4231)$$

since π avoids 312 and 213, $\pi^3 = 81726354$ avoids 213, and $\pi^5 = 24687531$ avoids 4231. The $\emptyset$ in positions 2 and 4 of the chain imply there are no avoidance restrictions placed on π^2 or π^4 .

In [2], we consider several chains, in particular those of the form $(312, 213 : \sigma)$ (i.e., unimodal permutations whose square avoids σ) and $(312, 321 : \sigma)$ for some σ of size 3. We also consider some cases of longer chains, also with patterns of length 3. Some of this work is expanded upon in [2, 8].

In this talk, we will discuss some of the results from [2] and consider some more recent results regarding chain avoidance as part of joint work with Christina Graves.

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A GRAPH ON BUMPLESS PIPE DREAMS

Joshua Arroyo

University of Florida

This talk is based on joint work with Adam Gregory and Prashreet Poudel

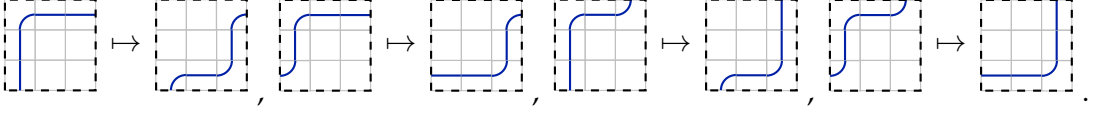
Bumpless pipe dreams are combinatorial objects used to study the Schubert polynomial associated to a permutation w [4]. The set of bumpless pipe dreams for w can be generated via certain local moves (called droops), giving rise to a natural graph structure. We initiate the study of this graph’s properties. In particular, we present a pattern-avoidance characterization of the permutations whose bumpless pipe dream graph is a tree. We begin by introducing the definitions needed to state our result.

A *reduced bumpless pipe dream* (BPD) of size n is a tiling of an $n \times n$ grid using the tiles



such that n pipes enter from the bottom, exit to the right, and no two pipes cross more than once [4]. Note that BPDs are in bijection with alternating sign matrices (ASMs) by replacing each  with a 1, each  with a -1 , and all other tiles by 0 [6]. Labeling the pipes in increasing order from left-to-right as they enter, one obtains the *associated permutation* by reading the pipe labels from top-to-bottom as they exit. Write $\text{BPD}(w)$ for the bumpless pipe dreams with associated permutation w .

A rectangle of tiles in a bumpless pipe dream B is called an *active region* if the top left corner is a $\begin{smallmatrix} \blacksquare \\ \nearrow \end{smallmatrix}$ -tile, the bottom right corner is a $\begin{smallmatrix} \blacksquare \\ \searrow \end{smallmatrix}$ -tile and there are no elbows in the region besides potentially on the corners. A *droop move* on an active region of some B in $\text{BPD}(w)$ is a reassignment of the tiles matching one of the following forms:



Unpictured pipes may pass through the region, but no elbow tiles appear within the region, as it is active. Note that a droop move maps B to another element of $\text{BPD}(w)$. A droop move is called *minimal* if it has no $\begin{smallmatrix} \blacksquare \\ \nearrow \end{smallmatrix}$ tiles in its active region. We call these *min-droops* for short. It follows from the work of Lam, Lee, and Shimozono [4] that $\text{BPD}(w)$ is connected via min-droops for any w .

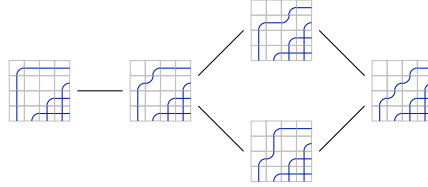
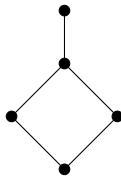


Figure 1: The BPD graph H_{1432} .

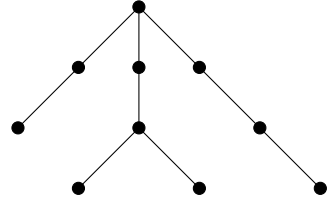
We define the *BPD graph* H_w of $w \in S_n$ to have vertex set $V(H_w) = \text{BPD}(w)$ and edges $E(H_w)$ given by $\{B_1, B_2\} \in E(H_w)$ if and only if B_1 and B_2 are related via a min-droop.

The BPD graph H_w has not been previously studied. Our main result is a complete characterization of which permutations w yield acyclic BPD graphs H_w .

Theorem 1. *The bumpless pipe dream graph H_w for w is a tree if and only if w avoids the fifteen patterns 1432, 12453, 12534, 13254, 13524, 14253, 14523, 24153, 31524, 35142, 214563, 216345, 354162, 461325, and 456132.*



BPD graph for $u = 1432 \in S_4$



BPD tree for $v = 321645 \in S_6$

We invite people to investigate other properties of the graphs H_w . For example, H_w is bipartite for every w and for the w that contain one of the fifteen patterns from Theorem 1, the girth of H_w is four. Moreover, we feel it would be interesting to study the diameter of H_w , perhaps via a statistic on the permutation w .

Motivation

We now provide context to motivate our result.

Schubert polynomials $\mathfrak{S}_w$ are a celebrated family of functions indexed by permutations w in S_n . They generalize Schur functions and represent cohomology classes of Schubert varieties in the complete flag variety [5]. There are several known combinatorial formulas for $\mathfrak{S}_w$. One of the earliest is the *pipe dream* (PD) model of Billey and Bergeron [2]. A more recent one is the *bumpless pipe dream* (BPD) model of Lam, Lee, and Shimozono [4]. In 2021, Gao and Huang provided a weight-preserving bijection between these two models [3]. Using their map, one can work interchangeably between the two models.

In 2025, Axelrod-Freed, Defant, Mularczyk, Nguyen, and Tung studied an analogous object to our BPD graph H_w , where they instead used the PD model [1]. They defined a graph on pipe dreams using a similar local generating move, called a “chute move”. They view chute moves as a directed edge and instead view their graphs on PDs as the Hasse diagram of a partial order on $\text{PD}(w)$. They prove that every chute move poset is a lattice. This is strikingly different than our situation. Indeed, if we view our BPD graphs as posets, we find that every H_w has a unique maximal element, but in general, there can be multiple minimal elements. In other words, the Gao–Huang bijection between BPD and PD does not provide a graph isomorphism, and the chute moves on PD provide a different structure than the droop moves on BPDs.

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WHEN DO THREE PREFIX-REVERSALS GENERATE ALL PERMUTATIONS? TAKING A CLOSER LOOK AT CUBIC PANCAKE GRAPHS

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This talk is based on joint work with M. Golubyatnikov, E. Konstantinova, N. Maslova and L. Nikiforov

For $2 \leq i \leq n$, let $r_i \in \text{Sym}_n$ be the *prefix reversal* that reverses the first i positions and fixes the remaining ones. Equivalently,

$$r_i = (1\ i)(2\ i-1) \cdots (\lfloor i/2 \rfloor\ \lceil i/2 \rceil).$$

The Cayley graph generated by all prefix reversals,

$$P_n = \text{Cay}(\text{Sym}_n, \{r_2, r_3, \dots, r_n\}),$$

is the classical *pancake graph*. If one fixes a triple $2 \leq k < m < n$, then

$$X_{n,m,k} = \text{Cay}(\text{Sym}_n, \{r_n, r_m, r_k\})$$

is a *cubic pancake graph*. Determining when $X_{n,m,k}$ is connected is equivalent to deciding when the three prefix reversals generate the full symmetric group.

Prefix reversals are among the most classical generators of the symmetric group: the involution r_i reverses the first i entries of a permutation, and the full set $\{r_2, \dots, r_n\}$ gives rise to the pancake graph P_n . The problem of determining which triples $\{r_n, r_m, r_k\}$ generate Sym_n lies at a natural intersection of algebraic combinatorics, permutation generation, and graph theory. Earlier work of Bass and Sudborough [2] gave basic necessary conditions and several infinite families of generating triples, while later work on the Big-3 pancake network connected these questions with Hamiltonicity and cyclic Gray codes [3].

The recent paper “Generating the symmetric group by three prefix reversals”¹ [1] gives a substantial partial answer to the general classification problem. Earlier work had already shown several necessary combinatorial conditions for a triple of distinct prefix reversals to generate Sym_n : one generator must be r_n ; one index must lie strictly between $\lfloor n/2 \rfloor$ and n ; at least one index must be even; at least one generator must be an odd permutation; and the three indices cannot have a common divisor greater than 1. Earlier papers also identified a short list of explicit generating families, including the Big-3 flips $\{r_n, r_{n-1}, r_{n-2}\}$.

The main contribution of the paper is a collection of new necessary and sufficient conditions for triples of the form $\{r_n, r_m, r_k\}$. Writing $l = n - m$, the paper first proves two general obstructions. For sufficiently large n , if $l \geq 2k + 1$, then $\langle r_n, r_{n-l}, r_k \rangle < \text{Sym}_n$. Moreover, if $n \geq 6$, $l \geq k + 1$, and $l \mid (n + 1)$, then again $\langle r_n, r_{n-l}, r_k \rangle < \text{Sym}_n$.

¹Submitted to the PP2025 Proceedings, but not presented at the conference.

These results show that many triples fail to generate because the action is forced to be intransitive.

The paper also completely classifies the four major infinite families in which one of the generators is r_2 , r_3 , r_{n-1} , or r_{n-2} . More precisely:

- $\langle r_n, r_m, r_2 \rangle = \text{Sym}_n$ exactly in the following cases: $m = n - 1$ when n is even; $m \in \{n - 1, n - 2\}$ when n is odd and $n \equiv 0, 2 \pmod{3}$; and $m \in \{n - 1, n - 2, n - 3\}$ when n is odd and $n \equiv 1 \pmod{3}$.
- $\langle r_n, r_m, r_3 \rangle = \text{Sym}_n$ exactly in the following cases: $m = n - 2$ when n is even; $m = n - 1$ when $n \equiv 3 \pmod{6}$; and $m \in \{n - 1, n - 3\}$ when $n \equiv 1, 5 \pmod{6}$.
- $\langle r_n, r_{n-1}, r_k \rangle = \text{Sym}_n$ if and only if one of the following holds: n is even and k is even; or $n \equiv 3 \pmod{4}$ and $2 \leq k \leq n - 2$; or $n \equiv 1 \pmod{4}$ and $k \equiv 2, 3 \pmod{4}$.
- $\langle r_n, r_{n-2}, r_k \rangle = \text{Sym}_n$ if and only if n and k have opposite parity.

Together, these results characterize all generating triples containing at least one of r_2 , r_3 , r_{n-1} , or r_{n-2} .

The paper also reports several computational patterns. Let $f(n)$ denote the number of pairs (m, k) with $2 \leq k < m < n$ such that $\langle r_n, r_m, r_k \rangle = \text{Sym}_n$. Based on exhaustive computations, the authors conjecture in Conjecture 1 that $f(n)$ has quadratic growth, with the leading term depending on $n \pmod{4}$:

$$f(n) = \begin{cases} \frac{1}{13}n^2 + O(n), & n \equiv 0 \pmod{4}, \\ \frac{1}{9}n^2 + O(n), & n \equiv 1 \pmod{4}, \\ \frac{1}{10}n^2 + O(n), & n \equiv 2 \pmod{4}, \\ \frac{1}{7}n^2 + O(n), & n \equiv 3 \pmod{4}. \end{cases}$$

Conjecture 2 in the paper asserts that every generating triple remains generating under the shift $(n, m, k) \mapsto (n + 4, m + 4, k + 4)$. We have a proof of a weaker result showing that if $\langle r_n, r_m, r_k \rangle = \text{Sym}_n$, then $\langle r_{n+4}, r_{m+4}, r_{k+4} \rangle$ acts transitively on $[n + 4]$.

Conjecture 3 predicts the analogous inheritance under $(n, m, k) \mapsto (n + 2, m + 2, k + 2)$ for $n \equiv 0, 1 \pmod{4}$.

Another central conjecture in the paper was the following non-generation criterion.

Conjecture 4. *If n is even and $m + k < n$, then $\langle r_n, r_m, r_k \rangle < \text{Sym}_n$. Moreover, if n is odd and $m + k < n - 1$, then $\langle r_n, r_m, r_k \rangle < \text{Sym}_n$.*

Another conjecture predicts a sharp formula for the maximum size of a row.

Conjecture 5. Let

$$F_k(n) = \{m : k < m < n, \langle r_n, r_m, r_k \rangle = \text{Sym}_n\}.$$

Then,

$$\max_{n>k+1} |F_k(n)| = \begin{cases} k+1, & k \text{ even} \\ \frac{k+1}{2}, & k \text{ odd} \end{cases}.$$

We report that **these two conjecture have now been proved** by the author. For the proof of Conjecture 4, we can write $l = n - m$. We show that one has non-generation whenever $l \geq k + 1$ in even degree and whenever $l \geq k + 2$ in odd degree. The proof organizes the points $\{1, \dots, n\}$ into columns modulo l and studies the action induced by the long reversals r_n and r_{n-l} together with the short reversal r_k . This yields either an invariant subset or a nontrivial block system, so the generated action is intransitive or imprimitive, and hence cannot be the full symmetric group. This new theorem explains the triangular forbidden region that is visible in the computational plots and substantially sharpens the paper's earlier obstruction $l \geq 2k + 1$. The proof of Conjecture 4 along with a construction of a particular value of n satisfying certain properties, establish Conjecture 5.

On the graph-theoretic side, the paper computes diameters, girths, and shortest cycles for all cubic pancake graphs arising from generating triples when $n = 7$ and $n = 8$, and tabulates extremal values for $4 \leq n \leq 10$. In the computed range, the minimal girth is 8 for every $n \geq 6$. The paper also verifies Hamiltonicity for the Big-3 pancake graph for $n = 4, 5, 6, 7, 8$, and for all eight generating triples when $n = 7$.

Beyond these explicit conjectures, the full classification of all generating triples r_n, r_m, r_k remains open, and on the graph-theoretic side it remains especially interesting to determine whether all connected cubic pancake graphs are Hamiltonian and, in particular, whether the Big-3 pancake graphs r_n, r_{n-1}, r_{n-2} admit cyclic Gray codes for every n , as conjectured by Sawada and Williams [3, Conjecture 4].

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This talk is based on joint work with Alvaro Cornejo, Reuven Hodges, Paul Horn, Nathan Lesnevich, Tyrrell McAllister

The symmetric group $\mathfrak{S}_n$ carries a number of natural partial orders, each with multiple equivalent characterizations suited to different contexts. Here we focus on the two most prominent, weak Bruhat order and strong Bruhat order, and give pattern-avoidance definitions of each. Our main results, Theorems 6 and 7, significantly improve the previous best-known bounds on the numbers of intervals in these orders.

Weak and strong Bruhat order have applications throughout algebraic and enumerative combinatorics. We will define both using a subword criterion later in this abstract, but remark here that these orders have rich mathematical histories. Strong order traces its origins to Schubert calculus, where it records containment of Schubert varieties [6, 5], and has since been studied representation theoretically [10], topologically [2], and combinatorially [1]. Weak order, which coarsens strong order, reflects containment of inversion sets. Through this lens, it connects to adjacent-swap sorting procedures [8] and can be understood as the one-skeleton of the permutohedron [11].

We define strong and weak Bruhat order in terms of expressions and subexpressions, which can be easily interpreted in the language of mesh patterns.

Definition 1. For $1 \leq i \leq n-1$, let $s_i \in \mathfrak{S}_n$ be the involution interchanging i and $i+1$. In one line notation, $s_i = 1\ 2 \cdots (i-1)\ (i+1)\ i\ (i+2) \cdots n$. We call s_i a *simple transposition*.

Any permutation π can be written as a product of simple transpositions, called an *expression* for π . We will call an expression *reduced* if its length, that is, its number of factors, is minimal. A permutation may admit many reduced expressions. For example, in $\mathfrak{S}_3$, $321 = s_1s_2s_1 = s_2s_1s_2$ and these are reduced because one can check exhaustively that it is impossible to express 321 as a product of fewer than 3 simple transpositions. The definition of weak Bruhat order in terms of expressions is as follows:

Definition 2. For $v, w \in \mathfrak{S}_n$, we say $v \leq_W w$ in *weak Bruhat order* if there exists a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ for w containing a prefix for v , that is, $k \leq \ell$ such that $v = s_{i_1} \cdots s_{i_k}$.

Unraveling this definition, we obtain the following characterization in terms of mesh patterns, as observed in [4]:

Theorem 3. For $v, w \in \mathfrak{S}_n$, w covers v in weak Bruhat order if and only if an occurrence of the mesh pattern  in v becomes an occurrence of the mesh pattern  in w .

We can similarly define strong Bruhat order in terms of expressions and give a mesh pattern interpretation of the covering relations:

Definition 4. For $v, w \in \mathfrak{S}_n$, we say $v \leq_S w$ in the *strong Bruhat order* if there exists a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ for w containing a subexpression for v , that is, $j_1 < \cdots < j_k$ such that $v = s_{i_{j_1}} \cdots s_{i_{j_k}}$.

Theorem 5. For $v, w \in \mathfrak{S}_n$, w covers v in strong Bruhat order if and only if an occurrence of the mesh pattern  in v becomes an occurrence of the mesh pattern  in w .

Given a poset $\leq$, it is natural to ask how many intervals it contains, that is, how many pairs $v \leq w$ are there. For Bruhat order, this was first studied by Hammett and Pittel [7]. They bounded the number of intervals in weak order on $\mathfrak{S}_n$ between $(n!)^2 \exp((-1 + o(1))n \log n)$ and $(n!)^2 \exp(o(1)n \log n)$. We improve both of these bounds and determine the precise asymptotics for weak order, proving that in some sense it lies exactly halfway between the bounds of Hammett and Pittel:

Theorem 6. Let $\mathcal{W}_n$ denote the set of intervals in weak Bruhat order on $\mathfrak{S}_n$. As $n \rightarrow \infty$,

$$|\mathcal{W}_n| = (n!)^2 \exp\left(-\frac{1}{2} + o(1)\right) n \log n.$$

Hammett and Pittel showed that the number of intervals in strong Bruhat order on $\mathfrak{S}_n$ is bounded between $(n!)^2(0.708)^n$ and $((n-1)!)^2$. We improve the lower bound from an exponential to a subexponential function:

Theorem 7. Let $\mathcal{B}_n$ denote the set of intervals in strong Bruhat order on $\mathfrak{S}_n$. As $n \rightarrow \infty$,

$$|\mathcal{B}_n| \geq (n!)^2 \exp(-(6 + o(1))\sqrt{n}(\log^{3/2} n)).$$

The lower bound in Theorem 6 is obtained by relating the number of elements smaller than $\sigma \in \mathfrak{S}_n$ in Bruhat order to the number of extensions of a poset $P(\sigma)$. Using known bounds on the longest increasing subsequence [9, Lemma 1.4], we restrict our attention to permutations whose longest increasing subsequence is not too long. We then use the Bochkov–Petrov lower inequality [3], which relates linear extensions of a poset to the number of antichains it contains, and consequently, the length of its longest chain via Mirsky’s theorem.

For the upper bound, we apply the Bochkov–Petrov upper inequality [3] to relate the number of linear extensions of $P(\sigma)$ to the lengths of chains of $P(\sigma)$. Via the Greene–Kleitman–Fomin chain functionals, lengths of chains are reflected by the Young diagram of the RSK shape of σ . Then, using Plancherel measure, we show that the probability of comparability in weak Bruhat order is bounded by an average of an explicit Young-diagram functional Ψ . The proof reduces to a uniform lower bound on Ψ , established via an inductive argument that peels away rows of the Young diagram.

To prove Theorem 7, we exhibit two large structured families of permutations of $\mathfrak{S}_n$ whose members are comparable in strong Bruhat order with probability $1 - o(1)$. Comparability is verified via the tableau (Gale-order) criterion, which reduces the required inequalities to a family of random-walk-type deviations controlled by concentration estimates. Specifically, fix $m = \lfloor 3\sqrt{n \log n} \rfloor$ and let $I_1 = \{1, 2, \dots, m\}$ and $I_2 = \{n - m + 1, n - m + 2, \dots, n\}$. We show that if $v(I_1) = I_1$, $v(I_2) = I_2$ and $w(I_1) = I_2$, $w(I_2) = I_1$, then $v <_S w$ with probability $1 - o(1)$.

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TWO WILF-EQUIVALENCES RELATED TO CHEBYSHEV POLYNOMIALS

Alexander Burstein

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This talk is based on joint work with Eric Egge and Sergi Elizalde

Let $[\ell_1, \dots, \ell_m]$ be the layered pattern with layer length sequence $(\ell_1, \dots, \ell_m)$. Let $F_{[\ell_1, \dots, \ell_m]}$ be the ordinary generating function for the class $\text{Av}(312, [\ell_1, \dots, \ell_m])$, and let $G_{[\ell_1, \dots, \ell_m]}$ be the ordinary generating function for the involution avoidance class

$\text{IAv}(3412, [\ell_1, \dots, \ell_m])$. Egge [1] conjectured that both $F_{[\ell_1, \dots, \ell_m]}$ and $G_{[\ell_1, \dots, \ell_m]}$ are symmetric in $\ell_1, \dots, \ell_m$; equivalently, all sets of patterns $(312, [\ell_{\pi(1)}, \dots, \ell_{\pi(m)}])$, where $\pi \in S_m$, are Wilf-equivalent, and all sets of patterns $(3412, [\ell_{\pi(1)}, \dots, \ell_{\pi(m)}])$, where $\pi \in S_m$, are involution-Wilf-equivalent.

We prove and generalize some special cases of these conjectures and refine these (involution-)Wilf-equivalences with respect to the distribution of some permutation statistics on those sets. Moreover, we prove that the total number of occurrences of $[\ell_1, \dots, \ell_m]$ in $\text{Av}_n(312)$ and in $\text{IAv}_n(3412)$ is symmetric in $\ell_1, \dots, \ell_m$, i.e. the number of occurrences of $[\ell_{\pi(1)}, \dots, \ell_{\pi(m)}]$ in $\text{Av}_n(312)$ and in $\text{IAv}_n(3412)$ is independent of $\pi \in S_m$.

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ON THE SCHUR POSITIVITY OF SMALL DESCENT ENUMERATORS FOR PATTERN AVOIDING SETS

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Let $\mathbb{P}$ be the set of positive integers. For $n \in \mathbb{P}$, let $[n] := \{1, 2, \dots, n\}$. A **small descent** of a word $w = w_1 w_2 \cdots w_n \in \mathbb{P}^n$ is a position $i \in [n-1]$ such that $w_i = w_{i+1} + 1$. Note that every small descent is a *descent* in the classical sense. Let $\text{SDES}(w)$ and $\text{DES}(w)$ be the sets of small descents and all descents of w , respectively. For a set $S \subseteq [n-1]$, let $F_{n,S}$ denote Gessel’s fundamental quasisymmetric function defined by

$$F_{n,S} := \sum_{S \subseteq T \subseteq [n-1]} M_{n,T},$$

where $M_{n,T}$ is the monomial quasisymmetric function. See [5, Chapter 7] for more background on symmetric and quasisymmetric functions. An elementary fact is

$$\sum_{w \in \mathfrak{S}_n} F_{n, \text{DES}(w)} = \sum_{\lambda \vdash n} f^\lambda s_\lambda,$$

where $\mathfrak{S}_n$ is the set of permutations of $[n]$, $\lambda \vdash n$ means λ is a partition of n , f^λ is the number of standard Young tableaux of shape λ , and s_λ is the Schur function of shape λ (see [5, Theorem 7.19.7]). This motivates the following question of Gessel and Reutenauer [2].

Question 1. For which subsets $W \subseteq \mathbb{P}^n$ is the function

$$Q_W := \sum_{w \in W} F_{n, \text{DES}(w)}$$

symmetric. Moreover, if Q_W is symmetric, is it Schur positive?

This question has been looked at extensively over the years: see [1],[4]. In this paper, we ask the same question, but for small descents.

Question 2. For which subsets $W \subseteq \mathbb{P}^n$ is the function

$$SQ_W := \sum_{w \in W} F_{n, \text{SDES}(w)}$$

symmetric. Moreover, if SQ_W is symmetric, is it Schur positive?

This quasisymmetric function for $W = \mathfrak{S}_n$ has also been looked at by Grinberg and Stanley [3].

These two questions can have quite different answers. For instance if W is the set of all permutations of the word 1123. Then Q_W is Schur positive (in fact, h -positive) whereas SQ_W is not:

$$Q_W = s_{2,1,1} + s_{2,2} + 2s_{3,1} + s_4 \quad SQ_W = s_{2,1,1} - s_{2,2} + 2s_{3,1} + 5s_4$$

On the other hand, if W is the set of 231-avoiding permutations of length 3 i.e. $W = \{123, 213, 132, 312, 321\}$, then SQ_W is Schur positive while Q_W is even not symmetric:

$$Q_W = F_{3, \{1,2\}} + 2F_{3, \{2\}} + F_{3, \{1\}} + F_{3, \emptyset} \quad SQ_W = s_{1,1,1} + s_{2,1} + 2s_3$$

In this talk, we provide 2 families of words W for which SQ_W is Schur positive: Π -avoiding permutations, where Π is any collection of patterns with no small descents, and the set of parking functions. Moreover, we will introduce two statistics for each that refines their expansions. First, for a set of patterns Π , let $\text{Av}_n(\Pi)$ be the set of permutation $\mathfrak{S}_n$ avoiding any pattern of Π .

Theorem 3. Let n be a positive integer and let Π be any set of patterns such that $\text{sdes}(\pi) = 0$ for all $\pi \in \Pi$. Then we have

$$SQ_{\text{Av}_n(\Pi)}(\mathbf{x}, q, t) := \sum_{\sigma \in \text{Av}_n(\Pi)} q^{\text{des}(\sigma) - \text{sdes}(\sigma)} t^{\text{asc}(\sigma)} F_{n, \text{SDES}(\sigma)} = \sum_{i=1}^n a_i^\Pi(q, t) s_{i, 1^{n-i}}.$$

where

$$a_n^\Pi(q, t) = \sum_{\substack{\sigma \in \text{Av}_n(\Pi) \\ \text{sdes}(\sigma)=0}} q^{\text{des}(\sigma)} t^{\text{asc}(\sigma)}$$

Moreover, for $\Pi = \{231\}$ or $\{312\}$, we show that the coefficients are enumerators of *Motzkin paths* (i.e. Dyck paths that also have flat steps). For a Motzkin path π , let $d(\pi)$ and $f(\pi)$ be the number of down steps and flat steps respectively.

Theorem 4. For all n , we have

$$a_n^{231}(q, t) = a_n^{312}(q, t) = \sum_{\pi \in \text{MW}_{n-1}} q^{d(\pi)} t^{d(\pi)+f(\pi)}$$

Recall that a **parking function** is a word α of positive integers such that for each i , there at least i entries of α less than or equal to i . Let PF_n be the set of parking functions of length n . For $\alpha \in \text{PF}_n$, let $\text{sasc}(\alpha)$ be the number of positions $i \in [n-1]$ such that $\alpha_i + 1 = \alpha_{i+1}$ (small ascents) and let $\text{tie}(\alpha)$ be the number of $i \in [n-1]$ such that $\alpha_i = \alpha_{i+1}$ (ties).

Theorem 5. For $n \geq 1$, we have

$$\text{SQ}_{\text{PF}_n}(\mathbf{x}, q, t) := \sum_{\alpha \in \text{PF}_n} q^{\text{sasc}(\alpha)} t^{\text{tie}(\alpha)} F_{n, \text{SDES}(\alpha)} = \sum_{i=1}^n (q + t + n - 2)^{i-1} s_{i, 1^{n-i}}.$$

It turns out that we can actually permute the statistics defining $\text{SQ}_{\text{PF}}(\mathbf{x}, q, t)$ and still retain the same function. The power sum expansion has a very pleasing form. For a partition $\lambda \vdash n$, let $z_\lambda = 1^{m_1} m_1! \cdots n^{m_n} m_n!$ where m_i is the number of parts of λ equal to i for each $i \in [n]$. Let $\mathbf{n} = (t-1) + (q-1) + n$, so $\text{SQ}_{\text{PF}_n}(\mathbf{x}, q, t) = \sum_{i=1}^n \mathbf{n}^{i-1} s_{i, 1^{n-i}}$.

Theorem 6. For all $n \geq 1$, we have

$$\text{SQ}_{\text{PF}_n}(\mathbf{x}, t) = \frac{1}{\mathbf{n} + 1} \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} \prod_{i=1}^{\ell(\lambda)} \left(\mathbf{n}^{\lambda_i} - (-1)^{\lambda_i} \right)$$

We will show why these Schur expansions all have a similar form by demonstrating that the equality of (quasi)symmetric functions is equivalent to the existence of a particular partitioning of $\text{Av}_n(\Pi)$ and PF_n , respectively. We end with a few future directions and open questions with the primary one being: “What other classes of permutations have Schur positive small descent enumerators?”

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SCHUR-POSITIVE STATISTICS ON 321-AVOIDING PERMUTATIONS AND OTHER CATALAN OBJECTS

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This talk is based on joint work with Yuval Roichman

The distribution of an integer-valued statistic st over a set $\mathcal{A}$ of combinatorial objects can be encoded by a polynomial $\sum_{a \in \mathcal{A}} x^{\text{st}(a)}$. To encode the distribution of a set-valued statistic $\text{St} : \mathcal{A} \rightarrow 2^{[n-1]}$, one often uses the quasisymmetric function

$$Q_{\text{St}}(\mathcal{A}) = \sum_{a \in \mathcal{A}} F_{n, \text{St}(a)},$$

where

$$F_{n,S} = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \dots x_{i_n}$$

is the fundamental quasisymmetric function indexed by $S \subseteq [n-1]$.

A commonly studied case is when $\mathcal{A}$ is a subset of the symmetric group $\mathcal{S}_n$ and St is the descent set defined by $\text{Des}(\pi_1 \pi_2 \dots \pi_n) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$. For example,

$$Q_{\text{Des}}(\mathcal{S}_3) = F_{3, \emptyset} + 2F_{3, \{1\}} + 2F_{3, \{2\}} + F_{3, \{1,2\}} = s_3 + 2s_{2,1} + s_{1,1,1},$$

where s_λ is the Schur function indexed by the partition λ .

A long-standing problem in algebraic combinatorics is the characterization of pairs $(\mathcal{A}, \text{St})$ for which $Q_{\text{St}}(\mathcal{A})$ is a symmetric function, and additionally, when it is *Schur positive*, meaning that all the coefficients in its expansion in the basis of Schur functions are nonnegative, as in the above example.

Classical examples of pairs for which $Q_{\text{St}}(\mathcal{A})$ is symmetric and Schur-positive include subsets $\mathcal{A} \subseteq \mathcal{S}_n$ that are closed under Knuth relations or under conjugation, where $\text{St} = \text{Des}$. Another example is when $\mathcal{A} = \text{SYT}(\lambda)$, the set of standard Young tableaux of a given shape λ , and St is the descent set on tableaux. Denoting this statistic again by Des , Gessel's fundamental theorem states that $Q_{\text{Des}}(\text{SYT}(\lambda)) = s_\lambda$.

In this work we consider several families of combinatorial objects enumerated by the Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$, and we describe natural set-valued maps on them that give rise to the same quasisymmetric functions. Our proofs combine new bijections with existing ones, for which we discover new statistic-preserving properties.

321-avoiding permutations. It is well known that $|\mathcal{S}_n(321)| = C_n$. As shown in [3, Prop. 2.6], it follows from properties of the Robinson–Schensted correspondence that

$$Q_{\text{Des}}(\mathcal{S}_n(321)) = \sum_{k=0}^{\lfloor n/2 \rfloor} |\text{SYT}(n-k, k)| s_{(n-k, k)}. \quad (1)$$

Denote by $\mathcal{S}_n^*(321)$ the set of indecomposable permutations in $\mathcal{S}_n(321)$, namely those for which there is no $i \in [n-1]$ such that the smallest i entries of π appear in the first i positions. It is known that $|\mathcal{S}_n^*(321)| = C_{n-1}$, and it is shown in [1, Thm. 1.2] that

$$Q_{\text{Des}}(\mathcal{S}_n^*(321)) = \sum_{k=1}^{\lfloor n/2 \rfloor} |\text{SYT}(n-k-1, k-1)| s_{(n-k, k)}. \quad (2)$$

Note that Equations (1) and (2) differ in the limits of the sum. Define $\overline{\text{Des}}(\pi)$ to be equal to $\text{Des}(\pi) \cup \{n\}$ if $\pi_n = n$, and to $\text{Des}(\pi)$ otherwise.

Transitive edge-colorings of the complete graph. Let K_n denote the complete graph on n vertices, and let $E(K_n)$ be its set of edges. An edge-coloring of K_n with m colors is a function $c : E(K_n) \rightarrow \{1, \dots, m\}$. A coloring is *transitive* if $c(i, k) \in \{c(i, j), c(j, k)\}$ for all $i < j < k$. A transitive coloring yields a partition of the edges of K_n , where each block consists of edges of one color. Partitions that arise in this way are called *transitive*. A transitive partition is called *maximal* if it has the largest possible number of (nonempty) blocks, which can be shown to be $n-1$. Let $\mathcal{P}_n$ be the set of maximal transitive partitions of the edges of K_n . It is known [2, Thm. 1.19] that $|\mathcal{P}_n| = C_{n-1}$. A *singleton* in $P \in \mathcal{P}_n$ is a block consisting of only one edge $(i, i+1)$. Let

$$\text{Sng}(P) = \{i \in [n-1] : \text{the edge } (i, i+1) \text{ is a singleton of } P\}.$$

An expression for $Q_{\text{Sng}}(\mathcal{P}_n)$ was found in [2, Thm. 1.24]. Comparing with Equation (2), one deduces that, surprisingly, $Q_{\text{Sng}}(\mathcal{P}_n) = Q_{\text{Des}}(\mathcal{S}_n^*(321))$. One of our results is a bijective proof of this equality, answering a question posed in [2].

Noncrossing alternating trees. A *noncrossing* tree with n vertices is a plane tree whose vertices lie on a circle and are labeled by $1, \dots, n$ in clockwise order, and whose edges are line segments that do not intersect in their interiors. A tree with labeled vertices is *alternating* if the vertices along any path in it form an alternating sequence. Let $\mathcal{N}_n$ denote the set of noncrossing alternating trees with n vertices. These were studied by Gelfand, Graev and Postnikov in the context of hypergeometric functions associated with positive roots. They showed that $|\mathcal{N}_n| = C_{n-1}$. For $N \in \mathcal{N}_n$, denote its set of *short edges* by

$$\text{Sho}(N) = \{i \in [n-1] : (i, i+1) \text{ is an edge of } N\}.$$

Binary trees. Let $\mathcal{B}_n$ denote the set of complete binary trees with n leaves. It is well known that $|\mathcal{B}_n| = C_{n-1}$. The leaves of $B \in \mathcal{B}_n$ have a natural ordering from left to right, defined recursively by first listing the leaves of the left subtree of the root of B , and then listing the leaves of the right subtree. We label the leaves of B with $1, 2, \dots, n$ from left to right. A *cherry* of B is a pair of leaves with a common parent. Let

$$\text{Che}(B) = \{i \in [n-1] : i \text{ and } i+1 \text{ form a cherry of } B\}.$$

Triangulations. Let $\mathcal{T}_n$ denote the set of *triangulations* of a regular n -gon. It is well known that $|\mathcal{T}_n| = C_{n-2}$. Label the vertices of the n -gon clockwise with $1, \dots, n$. An *ear* of a triangulation $T \in \mathcal{T}_n$ is a triangle in T whose vertices are adjacent vertices $(i, i+1, i+2)$ of the n -gon, where $i \in [n-2]$. Let

$$\text{Ear}(T) = \{i \in [n-2] : (i, i+1, i+2) \text{ is an ear of } T\}.$$

Lattice paths. Let $\mathcal{D}_n$ denote the set of *Dyck paths* of semilength n , defined as lattice paths from $(0,0)$ to (n,n) with steps $\mathbf{n} = (0,1)$ and $\mathbf{e} = (1,0)$ that stay weakly above the diagonal $y = x$. It is well known that $|\mathcal{D}_n| = C_n$. A *peak* of $D \in \mathcal{D}_n$ is a pair of adjacent steps $\mathbf{ne}$ in the path. Let

$$\text{Pea}(D) = \{i \in [2n-1] : \text{the } i\text{th and } (i+1)\text{st steps of } D \text{ form a peak}\}.$$

Define also $\text{Pea}_{\leq n}(D) = \text{Pea}(D) \cap [n]$, which is the set of peaks formed by the first $n+1$ steps of the path D . Denote by $\mathcal{D}_n^*$ the set of *elevated Dyck paths* in $\mathcal{D}_n$, which are those of the form $D = \mathbf{n}D'\mathbf{e}$, where $D' \in \mathcal{D}_{n-1}$. For $D \in \mathcal{D}_n$, define the statistic

$\Gamma(D) = \{i \in [n-1] : \text{the } i\text{th } \mathbf{e} \text{ step of } D \text{ is in the middle of an occurrence of } \mathbf{nee}\}$, and let $\bar{\Gamma}(D)$ be equal to $\Gamma(D) \cup \{n\}$ if D ends with $\mathbf{ne}$, and to $\Gamma(D)$ otherwise.

We can now state our main result.

Theorem 1. *There are explicit bijections proving that the following set-valued statistics on Catalan objects are equidistributed, and so the associated $Q_{\text{St}}(\mathcal{A})$ are equal and Schur-positive:*

- (a) $\overline{\text{Des}}$ on 321-avoiding permutations $\mathcal{S}_{n-1}(321)$,
- (b) Des on indecomposable 321-avoiding permutations $\mathcal{S}_n^*(321)$,
- (c) Sng on maximal transitive partitions $\mathcal{P}_n$,
- (d) Sho on noncrossing alternating trees $\mathcal{N}_n$,
- (e) Che on binary trees $\mathcal{B}_n$,
- (f) Ear on triangulations $\mathcal{T}_{n+1}$,
- (g) $\text{Pea}_{\leq n-1}$ on Dyck paths $\mathcal{D}_{n-1}$,
- (h) $\bar{\Gamma}$ on Dyck paths $\mathcal{D}_{n-1}$,
- (i) Γ on elevated Dyck paths $\mathcal{D}_n^*$.

For some of these families, we also consider generalizations enumerated by the Schröder numbers, and find that the corresponding set-valued statistics are still equidistributed.

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A STRETCHED EXPONENTIAL FOR $\text{Av}((12, 12), (312, 231))$

Andrew Elvey Price

CNRS, University of Tours

This talk is based on joint work with Juliette Schabanel, and Paul Th  venin

This abstract concerns pattern avoidance in 3-permutations. A 3-permutation of size n is simply a pair of permutations of $[n]$. Given a 3-permutation $\pi = (\pi_1, \pi_2)$, we denote by P_π the set of points

$$P_\pi = \{(j, \pi_1(j), \pi_2(j)) : j \in [n]\}.$$

Containment and avoidance in 3-permutation is defined as follows:

Definition 1. A 3-permutation $\pi = (\pi_1, \pi_2)$ of size n *contains* a 3-permutation $\tau = (\tau_1, \tau_2)$ of size $k < n$ if there exist integers $1 \leq x_1 < x_2 < \dots < x_k \leq n$ such that $\pi_1(x_i) < \pi_1(x_j)$ if and only if $\tau_1(i) < \tau_1(j)$ and $\pi_2(x_i) < \pi_2(x_j)$ if and only if $\tau_2(i) < \tau_2(j)$. In other words, π *contains* τ if there is a subset of P_π that is order-isomorphic to P_τ . Otherwise π *avoids* τ . The set of 3-permutations of length n avoiding τ is denoted by $\text{Av}_n(\tau)$.

Enumerating 3-permutations avoiding any single pattern appears extremely difficult: even the numbers $|\text{Av}_n((12, 12))|$ are unknown. These is equivalent to counting *intervals in the weak-Bruhat poset* (see [3, A007767]).

Recently Bonichon and Morel initiated the enumerative study of 3-permutations avoiding multiple patterns [3]. The first non-trivial cases are those of avoiding a pattern of size 2 and a pattern of size 3, for which there are three classes (up to symmetries). The first case has a surprisingly simple enumerative formula [1, 2]:

$$|\text{Av}_n((12, 12), (132, 312))| = (n + 1)^{n-1}.$$

The second case, $\text{Av}_n((12, 12), (123, 321))$ was shown to count intervals in the weak-Bruhat poset that are distributive lattices [3], although the exact enumeration is unknown (see [3, A190291]). The third and final case $\text{Av}_n((12, 12), (312, 231))$ is the subject of this work. These 3-permutations were recently shown to be in bijection with *bases of triangular percolation* [7], which we define below (see Figure 1). Bases of triangular percolation appear to have first been studied by Kirchner in 2017 [3, A295928] and later by Salo and Schabanel [6, 5].

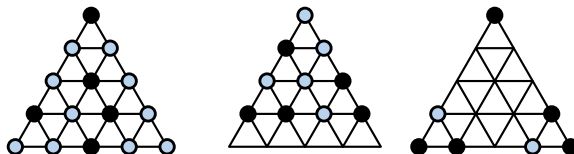


Figure 1: Left: A basis of size 5. Right: Two configurations that are not bases. The black points are the initially infected sites and the gray points are infected during the process.

Definition 2. In *triangular percolation* each of the $\frac{n(n+1)}{2}$ sites of a triangular grid of size n is initially either infected or uninfected. At each step, if any upward triangle of size 2 contains exactly 2 infected sites, the third becomes infected. An initial configuration with n infected sites is called a *basis* if the process starting with that configuration ends with all sites being infected.

Our first theorem characterises the counting function for $\text{Av}((12, 12), (312, 231))$, which is also the counting function for triangular bases, and shows that this generating function is *differentially algebraic*.

Theorem 3. Denoting $\Delta F = (zF(z))'$, there are unique series $T(z), I(z) \in \mathbb{Z}[[z]]$ satisfying the equations

$$T(z) = 1 + I(z)\Delta T(z)$$

and

$$T(z) = 1 + z + 3I(z)(\Delta T(z) - 1) - 3I(z)^2\Delta^2 T(z) + I(z)^3\Delta^3 T(z).$$

Moreover, $T(z)$ is the generating function for 3-permutations avoiding $(12, 12)$ and $(312, 231)$

$$T(z) = \sum_{n=0}^{\infty} |\text{Av}_n((12, 12), (312, 231))| z^n$$

We prove this theorem using a result of Salo and Schabanel that shows that any triangular basis can be cut into smaller triangular bases in one of three (non-disjoint) ways. The equation in Theorem 3 is then proven using an inclusion-exclusion argument to count these decompositions (see figure 2).

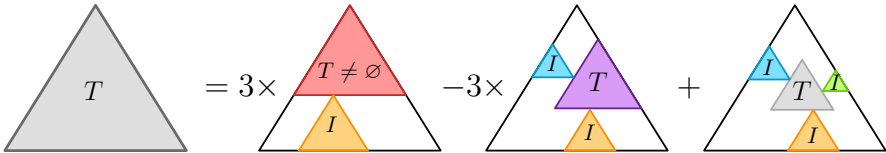


Figure 2: A schematic representation of the inclusion exclusion argument.

By analysing this equation, we find the asymptotic form of $|\text{Av}_n((12, 12), (312, 231))|$ up to some unknown constant C :

Theorem 4. There exists a constant $C > 0$ such that, as $n \rightarrow \infty$:

$$|\text{Av}_n((12, 12), (312, 231))| \sim Cn!e^{\sqrt{12n}}n^{5/12}.$$

Our empirical analysis of the exact terms $|\text{Av}_n((12, 12), (312, 231))|$ suggest that the constant $C \approx 0.00098107546$. A particularly interesting aspect of these asymptotics is the presence of a stretched exponential term $e^{\sqrt{12n}}$. To our knowledge this is the first proven case of stretched exponential asymptotics in the study of pattern avoiding permutations.

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THE MEX STATISTIC ON COMBINATORIAL STRUCTURES

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This talk is based on joint work with Antonio Bernini, Stefano Bilotta, and Giulio Cerbai

In Combinatorial Game Theory, the notion of mex of a subset of natural number is central in the analysis of impartial combinatorial games. Given $S \subseteq \mathbb{N}$, the **minimal excludant** of S , denoted $\text{mex}(S)$, is the minimum positive integer *not* belonging to S . Using the mex function, it is possible to compute the Sprague-Grundy numbers of positions in an impartial combinatorial game, which in turn allows to say if the position is winning or losing (among other things).

In recent years, the notion of mex has been extended to integer partitions by Andrews and Newman [2]. The main focus in this article, and in several subsequent ones, is however on number-theoretic properties mainly. Prior to Andrews and Newman's paper, Grabner and Knopfmacher [4] investigated the mex statistic on integer partitions under the name “smallest gap in partitions”, and also determined some genuinely combinatorial properties of it. In particular, it turns out that the generating function of integer partitions having mex m is rather easy to compute.

In this talk we describe a proposal to extend the notion of mex to several combinatorial structures, and we start the investigation of the the related combinatorial properties, especially from the point of view of enumeration. After providing a framework to

conveniently express the notion of mex inside a given combinatorial structure, we prove a general theorem (based on an inclusion/exclusion argument) which suggests how to compute the generating function of the objects having mex m , for any $m \in \mathbb{N}$. Our strategy turns out to be effective for several values of m (if assisted by a computer). We apply our result to several combinatorial structures, among which integer compositions, inversion sequences, Dyck paths, set partitions, and planar trees.

Without delving into details, to give just a glimpse of the kind of results that we are able to obtain, we consider here the case of inversion sequences. Recall that an *inversion sequence of size n* is a n -tuple $(x_1, x_2, \dots, x_n)$ of nonnegative integers satisfying the conditions $x_i \leq i - 1$, for all $i \leq n$. Inversion sequences are a useful encoding of permutations (see for instance [7]), whose combinatorial properties have extensively been investigated, especially in the context of patterns (see [1, 3, 8] to cite only a few of many papers dealing with inversion sequences). The mex of the inversion sequence $(x_1, x_2, \dots, x_n)$ is defined as the minimum integer m such that $x_i \neq m$ for all $i \leq n$. Our aim is to find information on the number of inversion sequences of size n having mex m . Let $\Gamma_m(x) = \sum_{n \geq 0} \gamma_{n,m} x^n$ be the generating functions of inversion sequences having mex m , with respect to the length.

Proposition 1. *The number $\gamma_{n,m}$ of inversion sequences of length n having mex m is 0 when $n < m$, 1 when $n = m$, and*

$$\sum_{k=1}^m (-1)^{k+1} (m - k + 1) S_{m, m-k+1} (n - k)!$$

when $n > m$, where $S_{n,k}$ denote the Stirling numbers of the second kind.

The number of inversion sequences of size $n \geq m$ having mex m is reported in the table below, for small values of m .

mex	number of inversion sequences of size $n \geq m$
$m = 1$	$\gamma_{n,1}^{\mathcal{IS}} = (n - 1)!$
$m = 2$	$\gamma_{n,2}^{\mathcal{IS}} = 2(n - 1)! - (n - 2)!$
$m = 3$	$\gamma_{n,3}^{\mathcal{IS}} = 3(n - 1)! - 6(n - 2)! + (n - 3)!$
$m = 4$	$\gamma_{n,4}^{\mathcal{IS}} = 4(n - 1)! - 18(n - 2)! + 14(n - 3)! - (n - 4)!$
$m = 5$	$\gamma_{n,5}^{\mathcal{IS}} = 5(n - 1)! - 40(n - 2)! + 75(n - 3)! - 30(n - 4)! + (n - 5)!$

Looking at the first values of $\gamma_{n,m}^{\mathcal{IS}}$, it turns out that the resulting table appears to match the table recorded as A056151 in [3] (with reversed rows). We have not been able to prove this striking coincidence.

We also remark that $\gamma_{n,1}$ is, by definition, the number of inversion sequences having no 1's. As we mentioned at the beginning of this section, we can encode permutations using inversion sequences. One way to do it is as follows: given a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ of length n , its inversion sequence $(x_1, x_2, \dots, x_n)$ is defined by setting

$x_i = |\{j \mid j < i, \pi_j > \pi_i\}|$ (i.e., x_i counts the number of elements of π to the left of π_i and forming an inversion with π_i). With this definition, it is easy to realize that an inversion sequence having mex 1 corresponds to a permutation avoiding the mesh pattern . As reported in [5], such permutations are indeed counted by $(n - 1)!$ (our mesh pattern corresponds to the complement of the mesh pattern labeled 8 in Table 3, page 12 of the mentioned paper).

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ON THE SCHUR-POSITIVITY OF SET PARTITIONS WITH A GIVEN NUMBER OF BLOCKS

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This talk is based on joint work with Eli Bagno

Introduction

The study of Schur-positivity for combinatorial families is a central topic in algebraic combinatorics, with a particularly well-developed theory in the setting of permutations (the necessary background is given below). Foundational work of Gessel [3]

introduced quasi-symmetric functions as generating functions for descent statistics, while Stanley [6] established their deep connections with symmetric functions and representation theory.

The extension of these ideas to set partitions is far less developed. For the special case of matchings, which can be viewed as set partitions with blocks with at most two elements, several independent approaches have established Schur-positivity for natural families. Marmor [4] introduced a Knuth-type equivalence on matchings and proved Schur-positivity for families defined by crossing statistics and related constraints. Adin and Roichman [2] and Yan et al. [8] studied descent statistics on matchings, showing that the associated quasi-symmetric generating functions are symmetric and Schur-positive. These results indicate that key structural features from the permutation setting - such as descent statistics and Knuth-like relations - extend to matchings.

Main results

In this work, we deal with a new descent function over set partitions, denoted $\text{Desing}(\pi)$, which is motivated by statistics appearing in Adin and Roichman [2]. This descent function counts the number of elements i , such that i is a singleton in a set partition π and $i + 1$ is not.

We prove Schur-positivity of some subsets of the set $\mathcal{PSet}(n)$ of set partitions:

Theorem 1. *We have the following equality:*

$$\sum_{\pi \in \mathcal{A}} \mathcal{F}_{n, \text{Desing}(\pi)} = \sum_k \mathcal{C}_{n,k} \mathcal{S}_{(n-k,k)},$$

where $\mathcal{A}$ is one of the subsets of $\mathcal{PSet}(n)$ appearing in the left column of Table 1, $\mathcal{F}_{n, \text{Desing}(\pi)}$ is the fundamental quasi-symmetric function with respect to the parameter Desing , and $\{\mathcal{C}_{n,k}\}$ is a sequence of coefficients appearing in the right column of Table 1. The different notations and coefficients appearing in Table 1 are as follows:

- $\mathcal{NC}(n)$ is the subset of non-crossing partitions in $\mathcal{PSet}(n)$.
- $\mathcal{PSet}(n, b)$ is the set of all set partitions of $[n]$ with b blocks.
- $\text{Bell}_{\geq 2}(n)$ are the associated Bell numbers, which count the set partitions of $[n]$ without singletons (see Sequence A000296 in OEIS [3]).
- r_i are the Riordan numbers, which count the non-crossing set partitions of $[n]$ without singletons (see Sequence A005043 in OEIS [3]).
- $S_{\geq 2}(n, k)$ are the associated Stirling numbers of the second kind, which count the set partitions of $[n]$ having b blocks, each of size at least 2 (see Sequence A008299 of OEIS [3]).

In our proof, we provide the appropriate mappings from the sets $\mathcal{A}$ to the set of standard Young tableaux of the shapes $(n - k, k)$, and also supply combinatorial meanings to the coefficients taking part in the decompositions.

Background: Symmetric functions, quasi-symmetric functions and Schur positivity

In this section, we follow [1]. Let $\mathbf{x} = \{x_1, x_2, \dots\}$ be a countably infinite set of commuting variables and consider the algebra of formal power series over $\mathbb{Q}$. A power

$\mathcal{A}$	$\mathcal{C}_{n,k}$
$\mathcal{PSet}(n)$	$\begin{cases} \sum_{i=2}^{n-1} \text{Bell}_{i \geq 2}(i) & k = 1 \\ \sum_{i=k}^{n-k} \text{Bell}_{i \geq 2}(i) & (k = 0) \text{ or } (k > 1). \end{cases}$
$\mathcal{NC}(n)$	$\begin{cases} \sum_{i=2}^{n-1} r_i & k = 1 \\ \sum_{i=k}^{n-k} r_i & (k = 0) \text{ or } (k > 1) \end{cases}$
$\mathcal{PSet}(n, b)$	$\begin{cases} \sum_{t=1}^{n-2} S_{\geq 2}(n-t, b-t) & k = 1 \\ \sum_{t=k}^{n-k} S_{\geq 2}(n-t, b-t) & (k = 0) \text{ or } (k > 1) \end{cases}$

Table 1: List of our main results

series $f \in \mathbb{Q}[[\mathbf{x}]]$ is called *symmetric* if it has a bounded degree and it is invariant under permutation of variables. Denote by Sym_n the vector space of symmetric functions, homogeneous of degree n . Bases for Sym_n are indexed by *partitions* λ of n (denoted $\lambda \vdash n$), or equivalently, by Young diagrams with n boxes.

The celebrated *Schur basis* of Sym_n is essential for our needs. Given a partition λ of n , a *standard Young tableau (SYT)* of shape λ is obtained by filling its boxes bijectively with the elements of $\{1, \dots, n\}$ so that rows and columns increase. In a *semistandard Young tableau (SSYT)* of shape λ , we demand that the entries are positive integers such that rows weakly increase and columns strictly increase. We write $\text{SYT}(\lambda)$ or $\text{SSYT}(\lambda)$ for the set of standard or semistandard Young tableaux of shape λ , respectively.

In the following definition, we define the notion of a *descent set* of an SYT T :

Definition 2. The *descent set* of a standard Young tableau T is defined to be:

$$\text{Des}(T) = \{i \mid i+1 \text{ is in a lower row than } i \text{ in } T\}.$$

The *Schur function* corresponding to a partition $\lambda \vdash n$ is: $s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} \prod_{i \in T} x_i$. For ex-

ample, the set $\text{SSYT}(3, 1)$ contains inter alia the following semistandard fillings of the shape $\lambda = \begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}$: $T_1 = \begin{smallmatrix} \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{2} \end{smallmatrix}$, $T_2 = \begin{smallmatrix} \boxed{1} & \boxed{2} & \boxed{2} \\ \boxed{2} \end{smallmatrix}$, $T_3 = \begin{smallmatrix} \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{4} \end{smallmatrix}$, which contribute

to $s_{(3,1)}$ the monomials $m_1 = x_1^3 x_2$, $m_2 = x_1 x_2^3$ and $m_3 = x_1 x_2 x_3 x_4$, respectively. It is well-known [6, Chap. 7] that the set $\{s_\lambda\}_{\lambda \vdash n}$ is a basis of Sym_n .

A symmetric function is called *Schur-positive* if all the coefficients in its expansion in the basis of Schur functions are nonnegative. Determining whether a given symmetric function is Schur-positive is a major problem in algebraic combinatorics [7].

Quasi-symmetric functions extend symmetric functions. Here is the formal definition:

Definition 3. A *quasi-symmetric function* is a formal power series $g \in \mathbb{Q}[[\mathbf{x}]]$ of bounded degree satisfying that any two of its monomials $x_{i_1}^{n_1} \dots x_{i_k}^{n_k}$ (where $i_1 < \dots < i_k$) and $x_{j_1}^{n_1} \dots x_{j_k}^{n_k}$ (where $j_1 < \dots < j_k$) have the same coefficient in g .

Clearly, every symmetric function is quasi-symmetric, but not conversely; for exam-

ple, $\sum_{i < j} x_i^2 x_j$ is quasi-symmetric but not symmetric.

The fundamental quasi-symmetric functions, which are defined now, serve as a basis for the vector space of homogeneous quasi-symmetric functions of degree n :

Definition 4. For each subset $D \subseteq [n - 1]$ define the *fundamental quasi-symmetric function* $\mathcal{F}_{n,D}(\mathbf{x}) := \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} x_{i_2} \cdots x_{i_n}.$

Let $\mathcal{B}$ be a (multi-)set of combinatorial objects, equipped with a *descent function* $\text{Des} : \mathcal{B} \rightarrow P([n - 1])$, which associates to each element $b \in \mathcal{B}$ a subset $\text{Des}(b) \subseteq [n - 1]$. Define the quasi-symmetric function $\mathcal{Q}_n(\mathcal{B}) := \sum_{b \in \mathcal{B}} m(b, \mathcal{B}) \mathcal{F}_{n, \text{Des}(b)}$, where $m(b, \mathcal{B})$ is the multiplicity of the element $b \in \mathcal{B}$.

Gessel showed that (see [6, Theorem 7.19.7]):

Theorem 5 (Gessel). *For every partition $\lambda \vdash n$, $\mathcal{Q}_n(\text{SYT}(\lambda)) = s_\lambda$.*

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This talk is based on joint work with Luc Lapointe and Michael D. Weiner

Charge and cocharge vectors

The charge statistic on words, introduced by Lascoux and Schützenberger [1], plays a central role in the combinatorics of symmetric functions. Our point of departure is a positional refinement of this statistic. Let $\sigma \in S_n$ be a permutation written in one-line notation, and let $i_k = \text{pos}_\sigma(k)$ denote the position of value k in σ . We define the *charge vector* $\mathbf{ch}(\sigma)$ and *cocharge vector* $\overline{\mathbf{ch}}(\sigma)$ by setting $\mathbf{ch}(\sigma)_{i_1} = \overline{\mathbf{ch}}(\sigma)_{i_1} = 0$, and for $k = 2, \dots, n$:

$$\mathbf{ch}(\sigma)_{i_k} = \mathbf{ch}(\sigma)_{i_{k-1}} + \mathbf{1}_{[i_k > i_{k-1}]}, \quad \overline{\mathbf{ch}}(\sigma)_{i_k} = \overline{\mathbf{ch}}(\sigma)_{i_{k-1}} + \mathbf{1}_{[i_k < i_{k-1}]}.$$

In other words, we scan values $1, 2, \dots, n$ in order: each right-to-left descent increments the running cocharge, and each left-to-right ascent increments the running charge, recording these counts at the position of the current value.

Example 1. For $\sigma = 37158264$, we have

$$\mathbf{ch}(\sigma) = (1, 3, 0, 2, 4, 1, 3, 2) \text{ and } \overline{\mathbf{ch}}(\sigma) = (1, 3, 0, 2, 3, 0, 2, 1).$$

A key structural identity follows directly from the definitions.

Lemma 2. For every $\sigma \in S_n$, $\overline{\mathbf{ch}}(\sigma) + \mathbf{ch}(\sigma) + \mathbf{1} = \sigma$, where $\mathbf{1} = (1, \dots, 1)$. In particular, $\text{ch}(\sigma) + \overline{\text{ch}}(\sigma) = \binom{n}{2}$, where $\text{ch}(\sigma)$ and $\overline{\text{ch}}(\sigma)$ denote the respective entry sums.

We also characterize which vectors arise as charge vectors.

Proposition 3. A vector $v \in \mathbb{N}_0^n$ equals $\mathbf{ch}(\sigma)$ for some $\sigma \in S_n$ if and only if: (i) the entries of v form a contiguous interval $\{0, 1, \dots, M\}$, and (ii) for every $1 \leq k \leq M$, the leftmost occurrence of $k - 1$ in v precedes the rightmost occurrence of k .

These conditions can be verified in $O(n)$ time, and the proof is constructive: there is a canonical permutation with any valid charge vector, obtained by placing values in decreasing order within each level set. We also establish how charge vectors interact with the standard symmetries: if σ^r is the reversal and σ^c the complement, then $\mathbf{ch}(\sigma) = \overline{\mathbf{ch}}(\sigma^r)^r$ and $\overline{\mathbf{ch}}(\sigma) + \mathbf{ch}(\sigma^c) = M \cdot \mathbf{1}$. For separable permutations, the charge vector records ascents precisely: $\max(\mathbf{ch}(\sigma)) = \text{asc}(\sigma)$ whenever σ is separable.

The charge-cocharge relation

The vectors $\mathbf{ch}$ and $\overline{\mathbf{ch}}$ combine naturally to define a relation on S_n .

Definition 4. For $\sigma, \tau \in S_n$, write $\sigma \rightarrow \tau$ if the entries of $w = \overline{\mathbf{ch}}(\sigma) + \mathbf{ch}(\tau)$ are distinct and $\max(w) - \min(w) = n - 1$.

Lemma 2 immediately gives $\sigma \rightarrow \sigma$, so the relation is reflexive. By the same lemma, if $\sigma \rightarrow \tau$ then w is a shifted permutation of $\{0, \dots, n - 1\}$, and the shift $m = \min(w)$ satisfies $nm = \mathbf{ch}(\tau) - \mathbf{ch}(\sigma)$. This yields:

Corollary 5. If $\sigma \rightarrow \tau$, then $\mathbf{ch}(\tau) \equiv \mathbf{ch}(\sigma) \pmod{n}$.

The relation is also compatible with the standard symmetries: $\sigma \rightarrow \tau$ implies $\tau^r \rightarrow \sigma^r$, $\tau^c \rightarrow \sigma^c$, and $\sigma^{rc} \rightarrow \tau^{rc}$. A more structural symmetry is the so-called *cyclage invariance*: if $\varrho = 23 \cdots n 1$ denotes the standard cyclic rotation, then $\sigma \rightarrow \tau$ implies $\sigma \circ \varrho \rightarrow \tau \circ \varrho$. This is proved by tracking how $\overline{\mathbf{ch}}$ and $\mathbf{ch}$ transform under right-composition with ϱ .

Theorem 6. The relation $\rightarrow$ on S_n has no directed cycles other than loops. That is,

$$\text{if } \sigma_1 \rightarrow \sigma_2 \rightarrow \cdots \rightarrow \sigma_\ell \rightarrow \sigma_1, \text{ then } \sigma_1 = \sigma_2 = \cdots = \sigma_\ell.$$

The proof uses induction on n . By cyclage, one may assume $\sigma_1(n) = 1$. Summing the shift values around the cycle forces each shift to be zero, which propagates the condition $\sigma_i(n) = 1$ to all i . Removing the 1 from each σ_i leads to a chain in S_{n-1} .

Charge classes and DAG structure

Corollary 5 shows that the relation $\rightarrow$ respects residue classes of $\mathbf{ch}$ modulo n . These classes partition S_n uniformly.

Proposition 7. The total charge statistic partitions S_n into n residue classes modulo n , each of size $(n - 1)!$.

Together with Theorem 6, this gives the main structural result.

Theorem 8. Within each charge residue class, the relation $\sigma \rightarrow \tau$ with $\sigma \neq \tau$ defines a directed acyclic graph (DAG) on $(n - 1)!$ vertices.

The n resulting DAGs share a rich combinatorial structure. The source elements (permutations with no incoming edges from distinct permutations) and sink elements (those with no outgoing edges) can be characterized explicitly. The DAG structure reflects how charge propagates under the arrow relation, and the level sets of total charge within each residue class form natural poset layers.

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RESTRICTED JACOBI PERMUTATIONS

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This talk is based on joint work with Alyssa G. Henke, Derek H. Stephens, Yongwei Yuan, and Yan Zhuang

Given a permutation $\pi \in \mathfrak{S}_n$ and $x \in [n]$, let $\rho_\pi(x)$ be the maximal consecutive subword of π consisting of letters immediately to the right of x which are all larger than x . We say that π is *Jacobi* if $|\rho_\pi(x)|$ is even for all $x \in [n]$. For example, the permutation $\pi = 4315624$ is not Jacobi because $|\rho_\pi(5)| = |\rho_\pi(2)| = 1$, but $\pi = 314652$ is Jacobi.

Jacobi permutations were introduced by Viennot [6] in the context of Jacobi elliptic functions. Viennot studied a sequence of polynomials appearing in the series expansions of the functions sn , cn , and dn , giving them a combinatorial interpretation as encoding the distribution of a certain statistic over Jacobi permutations. They are counted by the *Euler numbers* E_n defined by

$$\sec x + \tan x = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!},$$

which also count alternating permutations [1], André permutations [3], and simsun permutations [5]. Unlike these other families of permutations, very little seems to have been done with Jacobi permutations since their inception.

In this talk, we will survey our work [3] in which we conduct a systematic study of pattern avoidance in Jacobi permutations. We achieve a complete enumeration of Jacobi permutations avoiding a prescribed set of length 3 patterns, and in the case of a single pattern restriction, we obtain refined enumerations with respect to several permutation statistics: the number of ascents (or descents), the number of left-to-right minima, and the last letter. Furthermore, in analogy with doubly alternating permutations [4], we study *doubly Jacobi* permutations: permutations π such that both π and its inverse π^{-1} are Jacobi. Bijections involving certain subfamilies of binary trees and Dyck paths, as well as generating function techniques, play important roles in our proofs.

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PATTERN AVOIDANCE IN ALTERNATING SIGN RECTANGLES

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This talk is based on joint work with Matjaž Konvalinka (University of Ljubljana) and Svante Linusson (KTH Royal Institute of Technology, Stockholm)

When Robbins and Rumsey introduced the λ -determinant as a generalization of the ordinary determinant in the early 1980s [7], they discovered *alternating sign matrices*: An alternating sign matrix (ASM) is a square matrix with entries from $\{-1, 0, 1\}$ such that the nonzero entries in each row and in each column sum to 1 and alternate in sign. Clearly, permutation matrices are exactly those ASMs without any -1 . ASMs quickly gained popularity among combinatorialists due to their notoriously difficult enumeration. See [2] for an overview of the early history of ASMs. ASMs remain an active topic of research in enumerative combinatorics and related areas.

In contrast, there has been comparatively little research about pattern avoidance in ASMs with a few notable exceptions. We say that an ASM (*classically*) *avoids* a pattern π if there are no 1s in the ASM that appear in the same relative order as in the permutation matrix associated to π . In [4], Johansson and Linusson showed that ASMs avoiding a pattern of length 3 fall into two Wilf classes: those avoiding 132, 213, 231, or 312 are enumerated by the *large Schröder numbers* 1, 2, 6, 22, 90, 394, 1806, $\dots$, while those avoiding 123 or 321 are counted by the sequence 1, 2, 6, 23, 103, 514, 2785, $\dots$, for which no closed form is known. They further established a bijection between 132-avoiding ASMs and *Schröder paths*.

It is known that, for every pattern π , the number of π -avoiding permutation matrices grows at most exponentially due to a result by Marcus and Tados proving the Stanley–Wilf Conjecture. This behaviour does not extend to ASMs as the sequence of 123-avoiding ASMs grows super-exponentially. In fact, Bouvel et al. showed in [1] that there is only a finite set of patterns, namely $\{1, 12, 21, 132, 213, 231, 312, 2143, 2413, 3142, 3412\}$, for which the number of pattern avoiding ASMs grows at most exponentially.

In this work, we initiate the study of pattern avoidance on *alternating sign rectangles*.

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Definition 1. An *alternating sign rectangle* (ASR) of size $r \times k$ is an $r \times k$ matrix with entries from $\{-1, 0, 1\}$ such that in each row and in each column the nonzero entries alternate, and the leftmost nonzero entry in each row and the topmost nonzero entry in each column equals 1. If there are no nonzero entries in a row or column, the corresponding row or column is called *empty*. In addition, we require that rows (but not columns) have to be *balanced*, that is, nonempty rows should sum to 1.

See Figure 1 for an example. Every ASM is a square ASR such that each row and each column sums to 1.

$$\begin{array}{c}
 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\
 \text{(a)}
 \end{array}
 \qquad
 \begin{array}{c}
 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
 \text{(b)}
 \end{array}$$

Figure 1: Figure 1(a) shows a 5×6 ASR with four nonempty rows. Its extension in Figure 1(b) is a 312-avoiding ASM. Thus, the ASR in Figure 1(a) extendably avoids the pattern 312.

We study pattern avoidance of patterns of length 3 in ASRs in two different directions: *extendable pattern avoidance* and the asymptotics of classical avoidance.

Extendable pattern avoidance in ASRs

Definition 2. An ASR is said to *extendably avoid* a pattern π if there exists an extension of the ASR downwards and to the right to a square π -avoiding ASM. In other words, an ASR *extendably contains* a pattern if every extension contains the pattern. The notion of extended pattern avoidance in permutation matrices dates back to [3] and was further investigated in [5].

See Figure 1 for an example.

Our goal is to enumerate the number of ASRs with a given number of nonempty rows that extendably avoid patterns of length 3. To this end, let $S_{r,k,d}^\pi$ be the number of extendably π -avoiding ASRs of size $r \times k$ with d nonempty rows. We readily see that $S_{r,k,d}^\pi = 0$ if $r < d$ or $k < d$. Thus, we assume $r, k \geq d \geq 0$. Due to [4], we know that $S_{n,n,n}^\pi$ equals the n^{th} large Schröder number if $\pi \in \{312, 132, 213, 231\}$.

Theorem 3. We obtain a complete recursive description of the numbers $(S_{r,k,d}^\pi)_{r,k \geq d \geq 0}$ of extendably π -avoiding ASR of size $r \times k$ with d nonempty rows if $\pi \in \{312, 132, 213, 231\}$.

We omit the concrete systems of recurrence relations for brevity. In the case of extendably 312-avoiding ASRs, we even have a conjectural closed-form expression.

We also obtain bijective results for extendably 312-avoiding and 213-avoiding ASRs of special sizes such as the following, which is based on the bijection between 132-avoiding ASMs and Schröder paths presented in [4].

Theorem 4. *Let $r \geq 1$ be an integer. Then $S_{r,r+2,r}^{312}$ equals the number of lattice paths from $(0,0)$ to (r,r) with step set $\{(1,0), (0,1), (1,1)\}$ that do not cross the diagonal $y = x$ vertically, which are enumerated by [3, A268208].*

Asymptotics of classical pattern avoidance in ASRs

In this section, we consider ASRs of size $r \times k$ without empty rows, where $r \leq k$. Let $a_{r,k}^\pi$ denote the number of such ASRs that classically avoid π . We want to study the asymptotics of these numbers. To this end, we call an ASR without empty rows and empty columns *fundamental*. Let d_r^π denote the largest possible number of columns in a fundamental π -avoiding ASR with r rows and let f_r^π be the number of fundamental π -avoiding ASRs of size $r \times d_r^\pi$.

Theorem 5. *The number $a_{r,k}^\pi$ of π -avoiding ASRs of size $r \times k$ is asymptotically $\frac{f_r^\pi}{d_r^{\pi!}} k^{d_r^\pi}$.*

The following theorem provides expressions for the sequences d_r^π and f_r^π . Note that a horizontal reflection of an ASR proves that $a_{r,k}^{213} = a_{r,k}^{231}$, $a_{r,k}^{132} = a_{r,k}^{312}$, and $a_{r,k}^{123} = a_{r,k}^{321}$.

Theorem 6. *We have the following.*

- For $r \geq 1$, $d_r^{213} = 2r - 1$ and $f_r^{213} = C_{r-1}$, where $(C_r)_{r=0}^\infty$ is the sequence of Catalan numbers satisfying the recursion $C_{r+1} = \sum_{j=0}^r C_j C_{r-j}$.
- For $r \geq 2$, $d_r^{132} = 3r - 3$ and $f_r^{132} = D_{r-2}$, where $(D_r)_{r=0}^\infty$ is the sequence [3, A088716] satisfying the recursion $D_{r+1} = \sum_{j=0}^r (j+1) D_j D_{r-j}$.
- For $r \geq 1$, $d_r^{123} = 2r - 1$ and $f_r^{123} = E_{r-1}$, where $(E_r)_{r=0}^\infty$ is the sequence 1, 1, 3, 14, 86, 644, 5627, 55791, 615787, 7461509, 98209618, 1392498089, 21126311184, $\dots$ for which we have a combinatorial description.

We were not able to find a recursion for the sequence $(E_r)_{r=0}^\infty$, and we were only able to compute values up to $r = 15$. The sequence is currently not found in [3].

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EXCEDANCES AND DEFICIENCIES FOR INTEGER PARTITIONS

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Integer partitions are a classical topic in number theory dating back to work of Leibniz and Euler that continue to attract significant research. A partition of a positive integer n is an unordered collection $\lambda = (\lambda_1, \dots, \lambda_t)$ with $\sum_i \lambda_i = n$. For consistency, the parts λ_i are traditionally listed in nonincreasing order, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t$. The number of parts t is called the length of the partition.

In 2022, Blecher and Knopfmacher [1] considered partition fixed points, that is, λ with $\lambda_i = i$ for some index i . In resolving a conjecture of theirs, the speaker and James Sellers [4] made a connection between fixed points and an important partition statistic called the crank, named by Freeman Dyson in 1944 in hopes of a combinatorial understanding of a result by Ramanujan but not found until 1988 by George Andrews and Frank Garvan. The speaker [3] also considered partition fixed points more combinatorially, establishing a recurrence in the triangle of values $f(n, k)$, the number of partitions of n with fixed point $\lambda_k = k$. We are now working on related results for various restricted partitions [2].

In this talk, we continue a program of applying permutation statistics to partitions, here considering excedances and deficiencies ($\text{def}(\lambda) = |\{i : \lambda_i < i\}|$). For permutations, these statistics are closely related since $\text{exc}(\sigma) = \text{def}(\sigma^{-1})$ for $\sigma \in S_n$, so that, among all permutations of S_n , there are an equal number of excedances and deficiencies. Because partitions of n have varying lengths and allow repeated parts, the two statistics in this setting are more independent. For example, among the partitions of 4,

$$\left\{ \underset{+}{(4)}, \underset{+}{(3, 1)}, \underset{+}{(2, 2)}, \underset{+}{(2, 1, 1)}, \underset{-}{(1, 1, 1, 1)} \right\},$$

there are 4 excedances (marked +) and 6 deficiencies (marked –); the other two of the 12 total parts are fixed points (unmarked). We can explain heuristically why, among the partitions of n for $n \geq 3$, there are more deficiencies than excedances.

Let $e(n, k)$ be the number of partitions of n with k excedances. The resulting triangle of numbers has the same columns as the triangle of fixed point $f(n, k)$ values, just with different offsets. In fact, we can show $e(n, k) = f(n + k + 1, k + 1)$. Results from [3] then explain diagonal sums of the $e(n, k)$ triangle. There are several number triangles with the same columns at different offsets that document various types of partitions and their generalizations, such as overpartitions and unimodal compositions.

We have found less structure in the analogous $d(n, k)$ triangle of partition deficiencies. However, we can explain why the $k = 0$ column, the number of partitions of n with no deficiencies, matches the number of Rogers–Ramanujan partitions, an important type of restricted integer partition. We have also found a more involved formula for the $k = 1$ column.

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WELL-NESTED PERMUTATIONS AND SIGNED PERMUTATIONS

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This abstract concerns certain families of partial permutations defined in terms of a partial order and consecutive pattern avoidance. Let $\prec_A$ be the transitive closure of the relation on partial permutations that has $uBCAv \prec_A uCABv$ whenever u and v are words and $A, B, C \in \mathbb{Z}$ have $A < B < C$. Write $\overset{A}{\sim}$ for the reflexive and symmetric closure of this relation.

Definition 1. A partial permutation is *well-nested* if no element in its equivalence class under $\overset{A}{\sim}$ has any consecutive 321-patterns. A *well-nested family* is a union of equivalence classes under $\overset{A}{\sim}$ of well-nested partial permutations.

Example 2. The set $E = \{45362, 45623, 46253, 53462, 53624, 56234, 62453, 62534\}$ is a well-nested family which is minimal, i.e., a single equivalence class under $\overset{A}{\sim}$.

Well-nested families have a surprisingly rigid structure and a complete classification.

Theorem 3 ([4, 5]). *Any well-nested family of partial permutations is a graded poset under the relation $\prec_A$ and any interval in this poset has a unique minimal element and a unique maximal element. The minimal well-nested families in the symmetric group S_n are in bijection with its involutions $z = z^{-1} \in S_n$.*

Conjecture 4 ([4]). *Any minimal well-nested family is a lattice under $\prec_A$.*

Assume w is a partial permutation of length n . An index $i \in \{1, 2, \dots, n-1\}$ is a descent of w if $w_i > w_{i+1}$. The nested descent set of w is defined recursively by

$$\text{NDes}(w) = \begin{cases} \emptyset & \text{if } w \text{ has no descents,} \\ \{(w_i, w_{i+1})\} \sqcup \text{NDes}(w_1 \cdots w_{i-1} w_{i+2} \cdots w_n) & \text{if } i \text{ any descent.} \end{cases}$$

Likewise, define the nested residue of w to be the set

$$\text{NRes}(w) = \begin{cases} \{w_1, w_2, \dots, w_n\} & \text{if } w \text{ has no descents,} \\ \text{NRes}(w_1 \cdots w_{i-1} w_{i+2} \cdots w_n) & \text{if } i \text{ any descent.} \end{cases}$$

Theorem 5 ([5]). *If w is a well-nested partial permutation, then its nested descent set and nested residue are both well-defined. If v and w are both well-nested then $v \stackrel{A}{\sim} w$ if and only if $\text{NDes}(v) = \text{NDes}(w)$ and $\text{NRes}(v) = \text{NRes}(w)$.*

The preceding results have interesting extensions to type D.

Specifically, we now assume that w is an *even-signed partial permutation*, meaning a sequence of nonzero integers, of which an even number are negative, that becomes a partial permutation if all letters are replaced by their absolute values.

Define $\tilde{w} = \overline{w_1} w_2 w_3 \cdots w_n$ where $\overline{}$ indicates negation. Let $\prec_D$ be the transitive closure of the relation on partial permutations with $v \prec_D w$ if $v \prec_A w$ or $\tilde{v} \prec_A \tilde{w}$. Write $\stackrel{D}{\sim}$ for the reflexive, symmetric closure of this relation.

Definition 6. An even-signed partial permutation is *even-well-nested* if no element in its equivalence class under $\stackrel{D}{\sim}$ has any consecutive 321-patterns or consecutive 124 $\bar{3}$ -patterns. An *even-well-nested family* is a union of equivalence classes under $\stackrel{D}{\sim}$ of even-well-nested even-signed partial permutations.

In this context, w has a consecutive 124 $\bar{3}$ -pattern if $0 < w_i < w_{i+1} < -w_{i+3} < -w_{i+2}$ for some index i .

Example 7. The set $E = \{\bar{2}34\bar{1}, \bar{2}13\bar{4}, \bar{3}4\bar{2}1, 3\bar{4}\bar{2}1\}$ is an even-well-nested family which is minimal, i.e., a single equivalence class under $\stackrel{D}{\sim}$.

Theorem 8. *Any well-nested family of partial permutations is a graded poset under $\prec_D$ and any interval in this poset has a unique minimal element (but not always a unique maximum). The minimal well-nested families in the group W_n^D of even-signed permutations of*

$$[\pm n] = \{\pm 1, \pm 2, \dots, \pm n\}$$

are in bijection with all pairs (z, M) where $z = z^{-1} \in W_n^D$ and M is a symmetric noncrossing perfect matching on z 's negated points $\{i \in [\pm n] : z(i) = -i\}$.

Assume that $w = w_1 w_2 \cdots w_n$ is even-well-nested and write

$$\text{NRes}(w) = \{a_1 < \cdots < a_p < b_1 < \cdots < b_q\}$$

where $p, q \in \mathbb{Z}_{\geq 0}$ and $a_i < 0$ for all $i \in [p]$ and $0 < b_j$ for all $j \in [q]$.

Then define $\text{NDes}^\pm(w)$ by adding to the set $\{(|a|, b) : (a, b) \in \text{NDes}(w)\}$ the pairs

$(|a_i|, a_{i+1})$ for each odd $i \in [p-1]$ and $(|a_p|, b_1)$ if p is odd and $q > 0$ and $|a_p| > b_1$.

Let $\text{NRes}^\pm(w)$ be $\{b_2, \dots, b_q\}$ if $(|a_p|, b_1) \in \text{NDes}^\pm(w)$ and $\{b_1, \dots, b_q\}$ otherwise.

Theorem 9. *If v and w are both even-well-nested even-signed partial permutations, then $v \stackrel{D}{\sim} w$ if and only if both $\text{NDes}^\pm(v) = \text{NDes}^\pm(w)$ and $\text{NRes}^\pm(v) = \text{NRes}^\pm(w)$.*

Well-nested and even-well-nested families have applications to the geometry of orbit closures in complete flag varieties. Let G be a connected, simple algebraic group defined over the complex numbers. Assume G has a holomorphic automorphism θ .

Fix a Borel subgroup B preserved by θ . Then subgroup of fixed points

$$K = G^\theta = \{g \in G : g = \theta(g)\}$$

acts on the complete flag variety G/B with finitely many orbits [6, 7].

The Weyl group W of G indexes a family of (opposite) Schubert varieties $X_w = \overline{Bw_0wB}/B$ whose cohomology classes are a $\mathbb{Z}$ -basis for the cohomology ring $H^*(G/B)$. A general formula of Brion [1] gives the Schubert expansion of the cohomology class of the closure of each K -orbit in G/B .

In types A and D, Brion's formula has the following combinatorial interpretation.

Theorem 10 ([2, 3]). *Suppose $G = \text{GL}_n(\mathbb{C})$ and $\theta(g) = (g^{-1})^T$, so that $W = S_n$ and $K = \text{O}_n(\mathbb{C})$. Then the cohomology classes of the K -orbit closures $Y \subseteq G/B$ are*

$$[Y] = \sum_{w \in E} 2^{|\text{NDes}(w)|} [X_w]$$

where E ranges over all minimal well-nested families $E \subseteq S_n$.

Theorem 11. *Suppose $G = \text{SO}_{2n}(\mathbb{C})$ and θ is conjugation by the matrix $\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, so that $W = W_n^D$ and $K = \text{S}(\text{O}_n(\mathbb{C}) \times \text{O}_n(\mathbb{C}))$. Then the cohomology class of each K -orbit closure $Y \subseteq G/B$ has the form*

$$[Y] = \sum_{w \in E} 2^{|\text{NDes}^\pm(w)|} [X_w]$$

for a (not necessarily minimal) even-well-nested family $E \subseteq W_n^D$.

In this case, there is a closed formula (depending on the particular K -orbit closure) for the decomposition of the even-well-nested family E into minimal families. Much more generally, there are similar ways of realizing Brion's formula explicitly for all possibilities for $K = G^\theta$ when G is any classical group.

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AUTOMATIC DISCOVERY OF GENERATING TREES FOR PATTERN-AVOIDING INVERSION SEQUENCES

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This talk is based on joint work with Christian Bean and Jay Pantone

A size n inversion sequence is a word $\sigma_1\sigma_2\dots\sigma_n \in \mathbb{N}^*$ such that for all i , $\sigma_i < i$. Inversion sequences are in bijection with permutations, so there are $n!$ inversion sequences of size n . For a permutation π , the corresponding inversion sequence is $\sigma_1\sigma_2\dots\sigma_n$ where σ_i is the number of inversions ending at index i in π . For example, for the permutation 32514 the corresponding inversion sequence is 01031.

An inversion sequence σ can be represented on a plot on the Cartesian plane with points at (i, σ_i) for $i \in \{1, 2, \dots, n\}$. For example, the plot of inversion sequence 01031 is shown in Figure 1. Due to the restriction on values in an inversion sequence, all of the points in an inversion sequence must be below the dashed line in Figure 1.

We define a *Cayley permutation* to be a word $\pi \in \mathbb{N}^*$ that contains at least one occurrence of every value between 0 and the maximum value in π . Cayley permutations are not necessarily inversion sequences, for example 3210 and 1010 are Cayley permutations but not inversion sequences and 0002 is an inversion sequence but not a Cayley

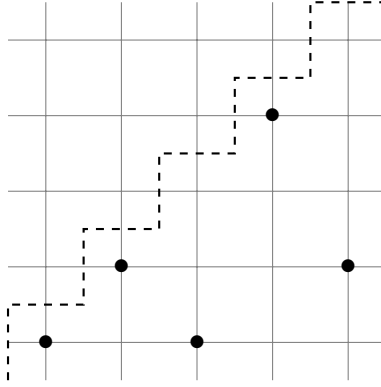


Figure 1: The inversion sequence 01031.

permutation. Cayley permutation avoidance and containment in inversion sequences is defined in the same way as pattern avoidance in permutations.

For a set of Cayley permutations B , we define the set of inversion sequences that avoid all Cayley permutations in B as $I(B)$. We call $I(B)$ an *inversion class* and B a *basis* of the inversion class. For example, the inversion class $I(010)$ is all inversion sequences that do not contain the Cayley permutation 010. The inversion sequences 0000, 0111, 0122 and 0123 are in $I(010)$ but 0010, 0110, 0020 and 0121 are not. A basis for an inversion class is not unique; for example $I(010, 0021) = I(10)$. For a Cayley permutation π , Testart [2] gave a characterisation of the minimal inversion sequences whose avoidance is equivalent to the avoidance of π for inversion sequences. This can be used to find a canonical representative of the basis of an inversion class.

Our aim is to enumerate inversion classes by algorithmically computing a generating function for the class. For all classes avoiding one or two patterns of size 3 a polynomial-time counting algorithm has been found such as a generating tree, nested summation, or an explicit or implicit generating function expression. See Testart [1] for an overview. A common method in literature of enumerating inversion classes, and one which we will use in this paper, is *generating trees*. A generating tree is a rooted, labelled tree such that the label of each node determines the labels of the children of that node.

In particular, a *combinatorial generating tree* is a generating tree such that each label of a node is an object in the combinatorial class. Figure 2 shows the first three levels of a combinatorial generating tree for inversion sequences in which the children of a node are obtained by adding a new rightmost value to the inversion sequence at that node. This method was defined by Testart [1] as a generating tree *growing on the right*. A combinatorial generating tree for an inversion class $I(B)$ is a rooted tree with each inversion sequence of size n in $I(B)$ the label of a node at level n .

Pantone [3] introduced the idea of making a *commitment* to which values will appear in inversion sequences when constructing them from left to right. We call these *configurations*, a pair (π, S) of an inversion sequence π with a set of commitments S to values which must appear. The set of all configurations forms a combinatorial class

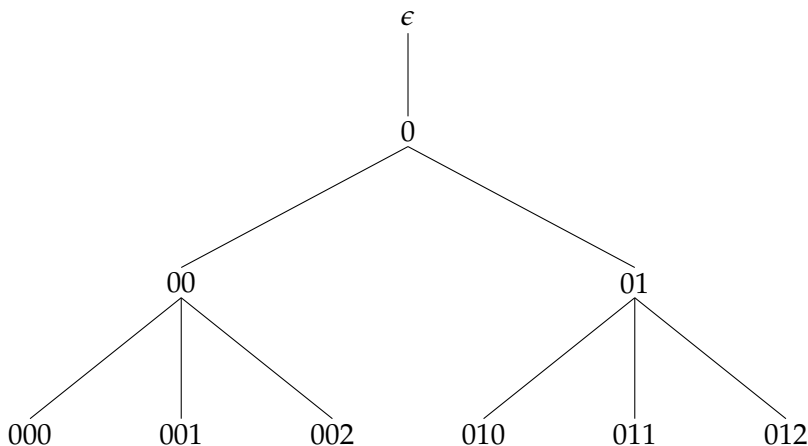


Figure 2: The first three levels of the combinatorial generating tree growing on the right for inversion sequences.

and in the combinatorial generating tree for this class the configurations without commitments are precisely the set of all inversion sequences. Figure 3 shows the first two levels of the combinatorial generating tree growing on the right for configurations.

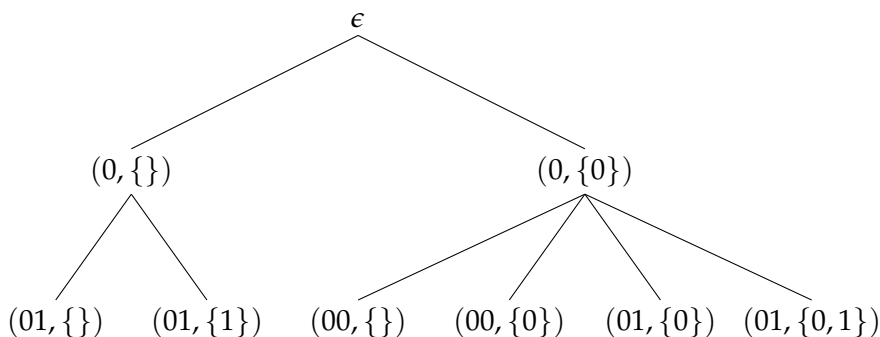


Figure 3: The first two levels of the combinatorial generating tree growing on the right for configurations.

As in the literature, instead of working with combinatorial generating trees directly we create a new generating tree which is isomorphic to it. We do this by creating new labels which more concisely describe the children of that node so can be used to represent multiple nodes on the combinatorial generating tree. When this new generating tree is finite, it can be used to find an algebraic generating function for the class. We have also proven conditions for when an inversion class can be represented by a finite generating tree in this way.

The labels we use have two parts. The first is a *tiling*, an adaptation the tilings defined by Albert, Bean, Claesson, Nadeau, Pantone, and Ulfarsson [4] in the context of permutations. Tilings keep track of the occurrences of basis patterns. The second is a *bounce* statistic, the difference between the height and length of an inversion sequence. This determines the new maximum values which can be added.

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COMPUTING WIN PROBABILITIES IN NIEDERMAIER’S DART GAME

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This talk is based on joint work with Christian Bean and Émile Nadeau

At the conference Permutation Patterns 2022, Andy Niedermaier introduced the following dart game. A group of players lines up single file at a dartboard. We assume all players are equally skilled and that all thrown darts hit the dartboard. Player 1 throws one dart and then moves to the back of the line. Player 2 then throws one dart. If their dart lands closer to the center of the dartboard than Player 1’s dart, they remain in the game and move to the back of the line. Otherwise, Player 2 is out. Then Player 3 throws one dart, remaining in the game and moving to the back of the line if and only if their dart hits closer than both of the darts thrown by Player 1 and Player 2. Play continues in this way: the player at the front of the line must hit the dartboard closer to the center than all previously thrown darts in order to remain in the game, moving to the back of the line if so, and otherwise being eliminated. The last remaining player is the winner.

What is the win probability of each player? In unpublished work, Niedermaier has calculated the win probabilities for each player in 2-, 3-, and 4-player games by setting up and solving certain systems of differential equations. Soon after, Elizalde [1] derived a recursive formula to compute these win probabilities, although the formula is computationally intensive and only permits the calculation of the win probabilities for 5-player games.

This talk will describe two new ways to compute win probabilities, each of which allows us to push past 5-player games. In the first, we use Niedermaier’s observation

that the relative distances of each dart to the center of the dartboard form a permutation describing the evolution of the game, and from this we derive counting formulas by enumerating a certain type of refined Dyck paths. The second approach uses a guess-and-check methodology to first conjecture solutions to Niedermaier's systems of differential equations and then verify those conjectures symbolically.

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SEPARABLE DERANGEMENTS

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This talk is based on joint work with Robert Dougherty-Bliss, Alejandro B. Galvan, and David Shuster

Abstract At the 2025 Permutation Patterns Pre-Conference Workshop, Vince Vatter asked how many separable permutations are also derangements, where the separable permutations are those that avoid 2413 and 3142, and derangements are those that have no fixed points [1]. Jay Pantone had computed the first 18 terms of the sequence using a brute-force algorithm and found that they were not in the OEIS. We present a system of bivariate generating functions which can be used to compute terms recursively – we have computed the first 1,000 terms in the sequence and have found no recurrence or guess for a closed form for the generating function. We conjecture that the generating function is not D-finite.

Our solution Let $T = \text{Av}(2413, 3142)$ and $T_n = \text{Av}_n(2413, 3142)$, and label the diagonals of the permutation from $-(n-1)$ to $(n-1)$, where the main diagonal has label zero. We define the following bivariate generating function:

$$f(x, q) = \sum a(n, k) x^n q^k \quad (3)$$

where $a(n, k) = \#\{\pi \in T_n : \pi \text{ has at least one point on diagonal } k\}$. This is *not* a refinement of the generating function for separable permutations as we typically think of refinement – there is no value of q we can substitute that would recover the original generating function. In fact, we are not counting each permutation in T_n an equal number of times. For example, the identity permutation $123 \dots n$ is counted only once (in the term $x^n q^0$), but the descending permutation $n(n-1)(n-2) \dots 21$ is counted n times, since each point lies on a different diagonal. In order to answer our original question, we simply subtract the term $a(n, 0)$ from the total number of separable permutations of length n .

Let $T^\oplus$ denote the separable permutations which are sum decomposable and $T^\ominus$ denote the separable permutations which are skew-sum decomposable. Then every $\pi \in T^\oplus$ can be decomposed as $\pi = \{1\} \oplus \sigma$ or $\pi = \tau \oplus \sigma$ for some $\tau \in T^\ominus$ and $\sigma \in T$. Similarly, $\pi \in T^\ominus$ can be decomposed as $\pi = \{1\} \ominus \sigma$ or $\pi = \tau \ominus \sigma$ from some $\tau \in T^\oplus$ and $\sigma \in T$.

Let $T(x, q)$ be our bivariate generating function for all permutations in T , $T^\oplus(x, q)$ be the restriction to permutations in $T^\oplus$, and $T^\ominus(x, q)$ be the restriction to permutations in $T^\ominus$. We will also need our non-refined univariate generating function for separable permutations, denote it by $T(x)$, and $T^\oplus(x)$ and $T^\ominus(x)$ for the restrictions to sum decomposable and skew-sum decomposable permutations, respectively.

We begin by analyzing the skew-sum decomposable permutations. Let τ be a permutation of length i with a point on diagonal j and let σ be a permutation of length k with a point on diagonal ℓ . Then, $\tau \ominus \sigma$ will have length $i + k$ and points on diagonals $j - k$ and $i + \ell$. Therefore, the pair $(x^i q^j, x^k q^\ell)$ needs to map to $x^{i+k} (q^{j-k} + q^{i+\ell})$ in our generating function. We accomplish this with the following operation:

$$T^\ominus(x, q) = (xq^0 + T^\oplus(x, q)) T\left(\frac{x}{q}\right) + (xq + T^\oplus(xq)) T(x, q). \quad (4)$$

Note that $T^\oplus(x, q)$ and $T(x, q)$ both denote the *bivariate* generating function as defined in (3), while $T\left(\frac{x}{q}\right)$ and $T^\oplus(xq)$ denote substituting $\frac{x}{q}$ and xq into the respective *univariate* generating functions. The first term accounts for diagonals which intersect τ , while the second term accounts for diagonals which intersect σ .

Now consider sum decomposable permutations. With τ and σ as before, $\tau \oplus \sigma$ once again has length $i + k$, but now has points on diagonals j and ℓ . Hence, the pair $(x^i q^j, x^k q^\ell)$ needs to map to $x^{i+k} (q^j + q^\ell)$. However, since we are only tracking whether a permutation has a point on a given diagonal and not how many, this overcounts permutations when $j = \ell$. We subtract the permutations we overcounted using the Hadamard product:

$$T^\oplus(x, q) = (xq^0 + T^\ominus(x, q)) T(x) + (x + T^\ominus(x)) T(x, q) - (xq^0 + T^\ominus(x, q)) \bigodot_q (T(x, q)) \quad (5)$$

where we define $\bigodot_q$ as

$$A(x, q) \bigodot_q B(x, q) = \sum_{m \in \mathbb{Z}} ([q^m] A(x, q) \cdot [q^m] B(x, q)) q^m.$$

Combining (4) and (5) gives us the following system of bivariate generating functions:

$$\begin{aligned} T(x, q) &= xq^0 + T^\oplus(x, q) + T^\ominus(x, q) \\ T^\ominus(x, q) &= (xq^0 + T^\oplus(x, q)) T\left(\frac{x}{q}\right) + (xq + T^\oplus(xq)) T(x, q) \\ T^\oplus(x, q) &= (xq^0 + T^\ominus(x, q)) T(x) + (x + T^\ominus(x)) T(x, q) - (xq^0 + T^\ominus(x, q)) \bigodot_q (T(x, q)) \end{aligned}$$

Using these generating functions we are able to quickly compute the number of separable derangements for large n recursively, and Jay Pantone has computed the first 1000 terms of the sequence. As far as we can tell, the sequence does not satisfy any nice recurrence, and we were not able to guess a closed form for the generating function. We conjecture that the sequence is not D-finite.

Acknowledgements We thank Jay Pantone for suggesting that we number the diagonals from $-(n-1)$ to $(n-1)$ and thereby simplifying some of our arguments and notation, as well as computing the first 1000 terms of the sequence.

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PATTERN AVOIDANCE ON TREES COMING FROM MODULI OF CURVES

Andrew Reimer-Berg

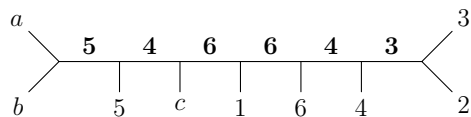
Colorado State University

Interested in a question in enumerative algebraic geometry concerning the intersection theory of moduli spaces of curves, Gillespie, Griffin, and Levinson [2] define a class of trivalent trees called *slide trees*. These trees are trivalent trees with leaves labeled by the alphabet $\{a, b, c, 1, 2, \dots, n\}$. Let $\underline{k}$ be a composition of n . (That is, a tuple $(k_1, \dots, k_n)$ of nonnegative integers with $\sum k_i = n$.) Then, [2] defines an algorithm that either labels the internal edges of a tree with the content $\underline{k}$, or fails to do so. We consider the set of trees that admit such a labeling.

Definition 1. Fix a composition $\underline{k}$ of n . Let $\text{Slide}^\omega(\underline{k})$ denote the set of trivalent trees with leaf labels $\{a, b, c, 1, 2, \dots, n\}$ with a and b adjacent, that admit a $\underline{k}$ -slide labeling.

Of particular interest are the slide trees that have caterpillar shape. That is, the trees whose non-leaf edges form a path. We denote the set of such trees by $\text{Cat}^\omega(\underline{k}) \subseteq \text{Slide}^\omega(\underline{k})$. Since the leaves a and b must be adjacent, they must appear on one end of the caterpillar or the other. We canonically consider the leaves a and b to together be the root of the tree, and always be drawn on the left side of the tree.

Example 2. An example of a caterpillar slide tree in $\text{Cat}^\omega(0, 0, 1, 2, 1, 2)$, along with the edge labels given by the slide labeling algorithm.



We may then consider the words obtained by reading the edge labels outwards from the root. For a given tree $T \in \text{Cat}^\omega(1, 1, \dots, 1)$, define the word $w(T)$ by reading the labels in the slide labeling of T from left to right. These words have many interesting pattern avoidance properties. In the case where $\underline{k} = (1, 1, \dots, 1)$, these words are all permutations. Gillespie, Griffin, and Levinson characterized the permutations obtained in this way in terms of vincular patterns.

Proposition 3 ([2], Proposition 6.2). *The set of words $\{w(T) : T \in \text{Cat}^\omega(1, 1, \dots, 1)\}$ are precisely the $23\text{--}1$ -avoiding permutations of length n , and in fact the words $w(T)$ are all distinct.*

Our results, which first appear in [1], generalize this to where $\underline{k}$ is no longer the all-1's composition. We first characterize which cases of $\underline{k}$ the words $w(T)$ can be characterized in terms of pattern avoidance conditions, as well as state said conditions. Then, we give the additional constraints for general $\underline{k}$.

Right-justified compositions

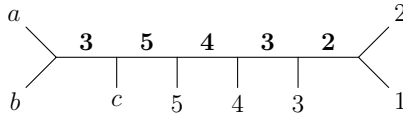
As it turns out, the set of compositions $\underline{k}$ for which the words $w(T)$ can be characterized in purely pattern avoidance terms is precisely the set of *right-justified compositions*.

Definition 4. A composition $\underline{k}$ of n is **right-justified** if there is some i for which $k_1 = \dots = k_i = 0$, and $k_j \neq 0$ for all $j > i$.

Whenever $\underline{k}$ is not the all-1's composition, we are no longer working with permutations, but instead with words. We define $\text{Av}_{\underline{k}}(\tau)$ to be the set of words with content $\underline{k}$ that avoid the pattern or collection of patterns τ . We thus state the class of words $w(T)$ for general right-justified compositions in terms of barred, vincular patterns on words.

Theorem 5. *Let $\underline{k}$ be a right-justified composition. Then, the set of words $\{w(T) : T \in \text{Cat}^\omega(\underline{k})\}$ are precisely the words in $\text{Av}_{\underline{k}}(2\text{--}1\text{--}2, 23\text{--}\bar{2}\text{--}1)$. Moreover, the words $w(T)$ are all distinct.*

Example 6. Consider the tree T below, drawn with its edge labels. Although the word $w(T) = 35432$ has the subword 352, which forms a $23\text{--}1$ pattern, $w(T)$ avoids the pattern $23\text{--}\bar{2}\text{--}1$.



Non-right-justified compositions

In the most general setting, the above pattern avoidance results still hold as necessary conditions, but are no longer sufficient. We define the additional quantities needed for our sufficient condition.

Definition 7. For a reverse-Catalan composition $\underline{k}$, define $z(i) := \#\{j > i \mid k_j = 0\}$.

Definition 8. For a word w with content $\underline{k}$, let $k(i, j)$ denote the number of j 's in the subword of w right of the rightmost i . Define the number of repeats larger than i right of the rightmost i by

$$\text{BigRep}_w(i) = \sum_{j=i+1}^n \max((k(i, j) - 1), 0).$$

Theorem 9. Let $\underline{k}$ be a composition of n . Then, $T \in \text{Cat}^\omega(\underline{k})$ if and only if $w(T) \in \text{Av}_{\underline{k}}(2-1-2, 23-\bar{2}-1)$ and

$$\text{BigRep}_w(i) \geq z(i)$$

for all i such that $k_i > 0$.

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THE RTLM SORTING ALGORITHM

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This talk is based on joint work with Lara Pudwell

Sorting permutations has a rich history which can be thought of as originating with Knuth [1]. We create and study an iterative sorting algorithm. The permutations sortable after k iterations form a non-classical pattern class with interesting enumeration and characterization properties.

Define the right-to-left-minima (Rtlm) sorting algorithm S applied to a permutation $\pi = \pi_1 \cdots \pi_n$ as follows:

1. Let $R(\pi)$ be the set of right-to-left-minima of π and let $R'(\pi) = \{\pi_1, \dots, \pi_n\} \setminus R(\pi)$.
2. Let $S(\pi)$ be the permutation obtained by listing all digits of $R(\pi)$ (in the same order as they appear in π) followed by all digits of $R'(\pi)$ (in the same order as they appear in π).

Example 1. Let $\pi = 243561$.

input	set of right-to-left-minima	output
$\pi = 243561$	$R(\pi) = \{1\}$	$S(\pi) = 124356$
$S(\pi) = 124356$	$R(\pi) = \{1, 2, 3, 5, 6\}$	$S^2(\pi) = 123564$
$S^2(\pi) = 123564$	$R(\pi) = \{1, 2, 3, 4\}$	$S^3(\pi) = 123456$

This algorithm will eventually sort any permutation with a sufficient number of applications.

Theorem 2. *The algorithm S will sort a permutation $\pi = \pi_1 \cdots \pi_n$ after at most $n - 1$ applications. Further, $n - 1$ is a sharp upper bound.*

We will utilize the following notation to more easily describe the sorting characteristics and capability of the Rtlm algorithm. Let $T(n, k)$ be the set of permutations of size n sortable via at most k applications of S and let $T^*(n, k)$ be the set of permutations of size n sortable via exactly k applications of S . Further, let $t_{n,k} = |T(n, k)|$ and let $t_{n,k}^* = |T^*(n, k)|$.

We can characterize and enumerate the permutations of size n that are sortable with a single application of S as follows.

Theorem 3. *For all n , we have $T(n, 1) = \text{Av}_n(321, 314\bar{2})$, and moreover $t_{n,1} = 2^{n-1}$ when $n > 0$.*

As is typical with sorting algorithms, it becomes more difficult to study permutations sortable after more than one application of the algorithm. However, we can enumerate the permutations of size n that are sorted after two applications of S as well as after three applications of S .

Theorem 4. *For all $n \geq 0$, we have $t_{n,2} = \frac{3^n - 2n + 3}{4}$.*

Theorem 5. *For all $n \geq 1$, we have $t_{n,3} = 2^{2n-3} - (n - 2)2^{n-2}$.*

Now consider this sorting problem from the opposite direction. Specifically, what are the permutations of size n that take the maximum number of applications of S to sort? We can characterize these permutations in terms of structure and also show they have a nice enumeration.

Theorem 6. *The permutations in $T^*(n, n - 1)$ are exactly the permutations that have the entry 2 appearing before the entry 1 and also avoid the patterns 123, 231, 312 formed by consecutive values within the permutation.*

Theorem 7. *For all positive integers n , we have that $t_{n,n-1}^*$ is given by A000111 in the OEIS [3], which enumerates down-up permutations [3].*

Taking this enumeration a step further, note that the Entringer numbers are a refinement of the Euler up-down numbers where $E_{n,k}$ refers to the number of alternating permutations of size $n + 1$ that begin with the entry $k + 1$.

A recursive formula for the Entringer numbers is given by: $E_{n,k} = E_{n,k+1} + E_{n-1,n-k}$.

We can show that permutations in $T^*(n, n-1)$ form an Entringer family either through a generating tree approach or by developing the recurrence directly as follows.

Theorem 8. *The permutations in $T^*(n, n-1)$ are an Entringer family where $E_{n,k}$ counts the number of permutations in $T^*(n+1, n)$ that have the 1 in position $k+1$.*

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BIJECTIONS BETWEEN PATTERN-AVOIDING DERANGEMENTS AND DESARRANGEMENTS

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This talk is based on joint work with Alyssa G. Henke and Yan Zhuang

A permutation is called a *desarrangement* if its first ascent is even. Desarrangements were introduced by Désarménien [2] for the purpose of studying derangements: permutations without fixed points. Désarménien proved that desarrangements are in bijection with derangements, and used desarrangements to give a combinatorial proof of the recurrence

$$d_n = nd_{n-1} + (-1)^n$$

where d_n is the number of derangements in the symmetric group $\mathfrak{S}_n$.

Robertson–Saracino–Zeilberger [5] and Mansour–Robertson [4] studied the distribution of fix (the number of fixed points) over all pattern avoidance classes $\mathfrak{S}_n(\Pi)$ where $\Pi \subseteq \mathfrak{S}_3$ and $|\Pi| \leq 3$. These specialize to results about pattern-avoiding derangements. More recently, Bsila et al. [1] studied pattern avoidance in desarrangements. Comparing these results on pattern-avoiding desarrangements with the earlier results on pattern-avoiding derangements led to the following.

Theorem 1 (Bsila et al. [1], Theorem 3.28). *Let $\Pi \subseteq \mathfrak{S}_3$ with $1 \leq |\Pi| \leq 3$. Define $d_n(\Pi)$ to be the number of desarrangements in $\mathfrak{S}_n(\Pi)$, and $\tilde{d}_n(\Pi)$ to be the number of derangements in $\mathfrak{S}_n(\Pi)$. Then $d_n(\Pi) = \tilde{d}_n(\Pi)$ for all $n \geq 0$ if and only if Π is one of the following: $\{132\}$, $\{132, 312\}$, $\{132, 321\}$, $\{213, 231\}$, $\{123, 132, 312\}$, $\{123, 213, 231\}$, $\{123, 312, 321\}$, $\{132, 312, 321\}$, $\{213, 231, 312\}$, or $\{213, 231, 321\}$.*

The usual bijection between derangements and desarrangements does not play nicely with pattern avoidance; finding bijective proofs for these equinumerosities was left as an open problem by Bsila et al.

Furthermore, just as derangements are permutations without fixed points, desarrangements are precisely the permutations without “pixed points”. To define pixed points, first note that every permutation π can be written uniquely as a concatenation $\pi = \iota\delta$ where ι is an increasing sequence and δ is a desarrangement. For example, $\pi = 46785213$ is the concatenation of $\iota = 467$ and $\delta = 85213$. Foata and Han [3] call this decomposition the *pixed factorization* of π and the letters of ι *pixed points*. Let $\text{pix}(\pi)$ denote the number of pixed points of π .

Bsila et al. did not refine their pattern avoidance results by the pix statistic, but noted that computational evidence suggests that pix and fix are equidistributed over $\mathfrak{S}_n(\Pi)$ for all but one Π from Theorem 1, the sole exception being $\Pi = \{132\}$.

Conjecture 2 (Bsila et al. [1], Conjecture 3.29). *Let $\Pi \subseteq \mathfrak{S}_3$ with $1 \leq |\Pi| \leq 3$. Then pix and fix are equidistributed over $\mathfrak{S}_n(\Pi)$ if and only if Π is one of the following: $\{132, 312\}$, $\{132, 321\}$, $\{213, 231\}$, $\{123, 132, 312\}$, $\{123, 213, 231\}$, $\{123, 312, 321\}$, $\{132, 312, 321\}$, $\{213, 231, 312\}$, or $\{213, 231, 321\}$.*

In our work, we resolve this conjecture in the affirmative by giving a bijective proof for every equidistribution listed above. Furthermore, we give a bijection showing that $d_n(132) = \tilde{d}_n(132)$ for all $n \geq 0$. Together, these also give a bijective proof for Theorem 1.

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In [1], Bóna and Pantone studied permutations that avoided all but one pattern of length $k + 1$ that began with a length k increasing subsequence. In this work, we extend this idea, studying similar permutation classes, where the index not part of the length k increasing subsequence is not necessarily $k + 1$.

We call these permutation classes almost distant patterns due to the similarity with distant patterns. As defined in [2], a permutation p contains a distant pattern $q = q_1 \dots q_i \square q_{i+1} \dots q_k$ if p contains a copy of $q_1 \dots q_n$ where q_i and q_{i+1} are not adjacent. It was shown in [2] that avoiding a distant pattern of length k (with a single box) is equivalent to avoiding a set of $k + 1$ classical patterns of length $k + 1$. If our excluded index is i , then avoiding our desired set of patterns is equivalent to avoiding all but one of the patterns that make up $12 \dots (i - 1) \square i \dots k$. We refer to this collection of patterns as $12 \dots (j - 1) \boxed{j} \dots k$.

Example 1. The almost distant pattern $12 \boxed{2} 3$ is equal to the collection of classical patterns $\{2314, 1234, 1243\}$.

We prove several Wilf-Equivalences related to almost distant patterns, and use those to provide some new bounds on the growth rates of distant patterns. The first theorem turns out to be a known result related to partial shuffles, first proved in [3], but proved here in a new manner.

Theorem 2. *Let $k \in \mathbb{Z}^+$, then the almost distant patterns $12 \dots (i - 1) \boxed{i} \dots k$ are Wilf-Equivalent for all $1 \leq i \leq k + 1$.*

Our second result follows as a corollary from the first, providing a new relatively tight bound on the growth rate of all monotone distant patterns.

Corollary 3. *Let $k \in \mathbb{Z}^+$. Then for all $0 \leq i \leq k$, $L(1 \dots i \square (i + 1) \dots k) \in [(k - 1)^2, (k - 1)^2 + 1]$.*

Our final result builds off of Theorem 2, and is the most interesting, giving an infinite family of infinite families of finite unbalanced Wilf-equivalences, or Wilf-equivalences where the two sides contain different lengths of patterns. There are very few of these known, with the first being found in 2014 in [1],

Theorem 4. *Let $k \geq 2$, $2 \leq j \leq k + 1$. Then there exists a finite set of permutations S such that $12 \dots (j - 1) \boxed{j} \dots k \subseteq S$ and $Av(12 \dots (j - 1) \boxed{j - 1} \dots k) = Av(S)$.*

For example, $j = 2$ gives an equivalence between k patterns of length $k + 1$, and k patterns of length $k + 1$ and one pattern of length $k + 2$.

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PERMUTATIONS WITH A FIXED NUMBER OF OCCURRENCES OF A PATTERN

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We investigate permutations with a fixed number of occurrences of a pattern, providing bounds for the asymptotics in certain cases, addressing a conjecture by Conway and Guttmann [3]. These bounds allow us to determine nonrationality and nonalgebraicity for the generating functions of these numbers for specific cases.

We begin with a definition.

Definition 1. Denote $S_{n,r}(q)$ to be the set of permutations on n entries which contain exactly r occurrences of the pattern q .

It is well known that $|S_n(321)| = |S_n(231)| = C_n$ with C_n the n th Catalan number. However it is not true that $|S_{n,1}(321)| = |S_{n,1}(231)|$.

Theorem 2 (Noonan 1996 [4]).

$$|S_{n,1}(321)| = \frac{3}{n} \binom{2n}{n+3}.$$

Theorem 3 (Bóna 1998 [1]).

$$|S_{n,1}(231)| = \binom{2n-3}{n-3}.$$

In fact, from these enumerations we can see that these numbers do not even have the same asymptotics.

$$\begin{aligned} |S_{n,1}(321)| &\sim \frac{12}{\sqrt{\pi}} \cdot \frac{4^n}{n^{3/2}} \\ |S_{n,1}(231)| &\sim \frac{2}{\sqrt{\pi}} \cdot \frac{4^n}{n^{1/2}}. \end{aligned}$$

Not much is known about $|S_{n,r}(q)|$ for general r , for most patterns q . Enumerating $|S_{n,r}(q)|$ in terms of n and r seems to be too difficult for most patterns. Instead, we aim to determine something about the asymptotics of $|S_{n,r}(q)|$.

The following is not difficult to prove.

Proposition 4. *Let q be a permutation and let $r \geq 1$. Then there are constants c_r, C_r so that for all n sufficiently large we have*

$$c_r |S_n(q)| \leq |S_{n,r}(q)| \leq C_r n^{2r} |S_n(q)|.$$

Note that this means that $|S_n(q)|$ and $|S_{n,r}(q)|$ always have the same exponential growth rate.

The following result due to Bóna [2] is the only other prior result of the type we are looking for that we know of.

Theorem 5 (Bóna 1997 [2]). *There is some constant C such that*

$$|S_{n,r}(231)| \sim C \frac{4^n}{n^{3/2-r}} \sim C n^r |S_n(231)|$$

Recently, Conway and Guttmann [3] collected a large amount of data and made the following conjecture.

Conjecture 6 (Conway and Guttmann 2023 [3]). *Let $q \in \{4321, 4312\}$. Then there is a constant K so that*

$$|S_{n,r}(q)| \sim C |S_n(q)|.$$

Let $q \in \{4123, 3412, 4231, 4213, 3142\}$. Then there is a constant C so that

$$|S_{n,r}(q)| \sim C n^r |S_n(q)|.$$

We prove the following results which specialize to address five out of the seven cases in the above conjecture.

Theorem 7. *Let p_0 be a permutation of any size, and let $r \geq 1$. Then there are constants c_r, C_r so that for all n sufficiently large we have*

$$c_r |S_n(321 \ominus p_0)| \leq |S_{n,r}(321 \ominus p_0)| \leq C_r |S_n(321 \ominus p_0)|.$$

Theorem 8. *Let q be a separable indecomposable permutation of length at least 2 which is of the form $q_1 \ominus \dots \ominus q_k$ for some $k \geq 2$ where all of the q_i are skew-indecomposable and $q_2, \dots, q_{k-1}$ are not 1. Then there are constants c and C such that*

$$c n^r |S_n(q)| \leq |S_{n,r}(q)| \leq C n^r |S_n(q)|.$$

Although these results do not give a full description of the asymptotics of these numbers, this level of precision does allow us to prove the following results about the nonrationality and nonalgebraicity of their generating functions in specific cases.

Theorem 9. Suppose $k \geq 3$. There is $c_{k,r}$ and $C_{k,r}$ such that

$$c_{k,r} \frac{(k-1)^{2n}}{n^{((k^2-2k)/2)}} \leq |S_{n,r}(k(k-1)\dots 1)| \leq C_{k,r} \frac{(k-1)^{2n}}{n^{((k^2-2k)/2)}}.$$

Let $S_r(z) = \sum_{n \geq 0} |S_{n,r}(q)| z^n$. Then:

- If k is odd, then $S_r(z)$ is not rational.
- If k is even, then $S_r(z)$ is not algebraic.

Theorem 10. Suppose $r \geq 1$. Then there is c_k and C_k such that

$$c_k \frac{(k-1)^{2n}}{n^{((k^2-2k)/2)-r}} \leq |S_{n,r}(23\dots k1)| \leq C_k \frac{(k-1)^{2n}}{n^{((k^2-2k)/2)-r}}.$$

Let $S_r(z) = \sum_{n \geq 0} |S_{n,r}(23\dots k1)| z^n$. Then:

- If k is odd, then $S_r(z)$ is not rational.
- If k is even and $r < \frac{k^2-2k}{2}$, then $S_r(z)$ is not algebraic.

Theorem 11. Suppose $r \geq 1$. Then there is c and C such that

$$c \frac{9^n}{n^{4-r}} \leq |S_{n,r}(3412)| \leq C \frac{9^n}{n^{4-r}}.$$

Let $S(z) = \sum_{n \geq 0} |S_{n,r}(3412)| z^n$. If $r < 4$, then $S(z)$ is not algebraic.

Theorem 12. Suppose $r \geq 1$. Then there is c and C such that

$$c \frac{8^n}{n^{5/2-r}} \leq |S_{n,r}(4213)| \leq C \frac{8^n}{n^{5/2-r}}.$$

Let $S(z) = \sum_{n \geq 0} |S_{n,r}(4213)| z^n$. Then $S(z)$ is not rational.

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POLYNOMIAL TIME ENUMERATION OF t -STACK-SORTABLE PERMUTATIONS ENDING IN THEIR LEAST ENTRY

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Background

In 1990, West [4] introduced the stack-sorting map s from the set of permutations to itself. For each permutation π , the map iteratively pushes entries of π into a stack so long as either the stack is empty or the top element of the stack is greater than the next entry of π . If the next entry cannot be pushed into the stack, then the top element of the stack is popped out and appended to the output permutation. The process continues until all elements of π are appended to the output permutation. Note that the maximal unsorted entry of π is always sent to its correct position when s is applied to π , so $s^{|\pi|-1}(\pi)$ must be increasing. Let $\langle \pi \rangle$ denote the minimum integer k for which $s^k(\pi)$ is increasing. A permutation π is said to be t -stack-sortable if and only if $\langle \pi \rangle \leq t$.

The set of all t -stack-sortable permutations in S_n is denoted by $\mathcal{W}_t(n)$. Knuth [2] showed that $|\mathcal{W}_1(n)| = C_n$, where C_n denotes the n th Catalan number. Then, an explicit formula for $|\mathcal{W}_2(n)|$ was conjectured by West and proved by Zeilberger [5].

$$|\mathcal{W}_2(n)| = \frac{2}{(n+1)(2n+1)} \binom{3n}{n}.$$

In 2020, Defant [1] gave the first polynomial-time algorithm for counting $|\mathcal{W}_3(n)|$; for $t \geq 4$, the enumeration of $|\mathcal{W}_t(n)|$ in polynomial time remains an open problem.

Main Results

We study the dynamics of stack-sorting on permutations ending in their least entry. Let $S'_n := \{\pi 0 \mid \pi \in S_n\}$ where $\pi 0$ denotes the concatenation of π and 0. The set of all t -stack-sortable permutations in S'_n is denoted by $|\mathcal{W}'_t(n)|$. Our main result is Theorem 1.

Theorem 1. *For fixed $t \in \mathbb{N}$, the value $|\mathcal{W}'_t(n)|$ is computable in $O(n^{3t+1})$ time.*

We prove Theorem 1 by introducing a new combinatorial object known as the stack-sorting tableau T_π , which records the evolution of $\pi \in S'_n$ under repeated applications of s . To justify the relevance of studying stack-sorting on S'_n , we also establish a relationship in Theorem 2 between $\langle \pi \rangle$ and $\langle \pi^i \rangle$ for all $\pi \in S_n$ and $i \in [n]$, where π^i denotes the permutation obtained by taking the first i entries of π and removing any entries less than π_i . Note that the filtering process leaves π^i as a permutation whose last entry is its least.

Theorem 2. *For all $\pi \in S_n$,*

$$\langle \pi \rangle = \max_{i \in [n]} \langle \pi^i \rangle.$$

Additionally, we make a conjecture on the value of $|\mathcal{W}'_2(n)|$.

Conjecture 3. *For all $n \in \mathbb{N}$, the value $|\mathcal{W}'_2(n)|$ is the n th Motzkin number [3].*

For more information, see <https://arxiv.org/abs/2604.10779>.

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STATISTICS AND PATTERN AVOIDANCE ON DESARRANGEMENTS

Yan Zhuang

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This talk is based on joint work with Chadi Bsila, Caroline E. Cox, Anna S. Hugo, and Lindsey A. Styron

A permutation is called a *desarrangement* if its first ascent is even. Desarrangements were introduced by Désarménien [2] for the purpose of studying derangements: permutations without fixed points. Désarménien proved that desarrangements are in bijection with derangements, and used desarrangements to give a combinatorial proof of the recurrence

$$d_n = nd_{n-1} + (-1)^n$$

where d_n is the number of derangements in the symmetric group $\mathfrak{S}_n$.

Most of the literature on desarrangements focus on their connection to derangements. Our work [1] was motivated by the desire to further study desarrangements for their own sake. We focus on two themes: the refined enumeration of desarrangements by permutation statistics, and pattern avoidance in desarrangements.

The permutation statistics that we consider include the number of descents, peaks, valleys, double ascents, and double descents. Taking $\mathfrak{D}_n$ to be the set of desarrangements in $\mathfrak{S}_n$, we obtain a generating function formula for the distribution of each of

these statistics over $\mathfrak{D}_n$. For example, the exponential generating function for counting desarrangements by double ascents is found to be

$$\sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{D}_n} t^{\text{dasc}(\pi)} \frac{x^n}{n!} = \frac{e^{-x} \left((2-t)\alpha \cosh\left(\frac{\alpha x}{2}\right) - t(1-t) \sinh\left(\frac{\alpha x}{2}\right) \right) + (1-t)\alpha e^{(1-t)x/2}}{(3-2t) \left(\alpha \cosh\left(\frac{\alpha x}{2}\right) - (1+t) \sinh\left(\frac{\alpha x}{2}\right) \right)}$$

where $\alpha = \sqrt{(t+3)(t-1)}$. We also study the joint distribution of peaks and valleys over $\mathfrak{D}_n$, as well as the joint distribution of descents and “pixed points” over the full symmetric group $\mathfrak{S}_n$. These results are obtained using (a generalization of) Gessel’s “run theorem” [3, 4], a reciprocity formula involving noncommutative symmetric functions which provides a systematic method for counting permutations by statistics expressible in terms of runs.

Given a set Π of patterns, let $d_n(\Pi)$ denote the number of desarrangements in $\mathfrak{D}_n$ which avoid every pattern in Π . Using a mixture of combinatorial and generating function methods, we determine $d_n(\Pi)$ for all $\Pi \subseteq \mathfrak{S}_3$. For example, for the singleton pattern sets, we find that

$$d_n(321) = C_{n-1}, \quad d_n(213) = d_n(312) = \sum_{k=1}^{n-1} (-1)^{n-k-1} C_k,$$

$$\text{and } d_n(132) = d_n(231) = d_n(123) = F_{n+1} \quad \text{for all } n \geq 2;$$

here C_n is the n th Catalan number and F_n is the n th Fine number. Our results also include new interpretations of the Jacobsthal and Fibonacci numbers as counting certain families of pattern-avoiding desarrangements.

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— Posters —

A JUMP IN THE SPEED OF HEREDITARY GRAPH CLASSES

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This poster is based on joint work with Bertalan Bodor and James Hanson

A hereditary graph class is a class of graphs closed under induced subgraph, and is the analogue of a permutation class in the graph setting. We consider the labeled enumeration of such classes, as given by following definition.

Definition 1. Let $\mathcal{C}$ be the set of isomorphism types of a hereditary graph class, and let $\mathcal{C}_n$ be the set of realizations of the isomorphism types in $\mathcal{C}$ with n vertices on the vertex set $[n]$. The *speed* of $\mathcal{C}$ is the function $f: \mathbb{N} \rightarrow \mathbb{N}$ given by $n \mapsto |\mathcal{C}_n|$.

The possible speeds of hereditary graph classes and related structural information have been extensively studied, and are surveyed, e.g., in [3]. In particular, there is the following result describing the possible speeds of hereditary graph classes. The resulting spectrum has four bands of growth with gaps between them.

Theorem 2 ([1, Theorem 1]). *Let $\mathcal{C}$ be a hereditary graph class and let $f_{\mathcal{C}}$ be its speed. Then one of the following holds.*

1. *For all sufficiently large n , we have $f_{\mathcal{C}}(n) \in \{0, 1, 2\}$.*
2. *$f_{\mathcal{C}}(n) = \Theta(1)n^k$ for some positive integer k .*
3. *For some positive c_1, c_2 , we have $c_1^n < f_{\mathcal{C}}(n) < c_2^n$.*
4. *For some $c > 0$, we have $n^{cn} \leq f_{\mathcal{C}}(n)$.*

Our result identifies a new gap towards the bottom of the third range, corresponding the property of *bounded shrub-depth*. We refer to [4] for a definition of shrub-depth, and simply remark that graphs in classes of bounded shrub-depth are structurally close to trees of bounded height.

Theorem 3 ([4, Theorem 1.2]). *Let $\mathcal{C}$ be a hereditary graph class and let $f_{\mathcal{C}}$ be its speed. Then one of the following holds.*

1. *$\mathcal{C}$ has bounded shrub-depth and $f_{\mathcal{C}}(n) < \frac{n!}{c^n}$ for every $c > 0$, for sufficiently large n .*
2. *$\mathcal{C}$ has unbounded shrub-depth, and $f_{\mathcal{C}}(n) > \frac{n!}{10^n}$, for sufficiently large n .*

In [2, Theorem 10], it is shown that for every $c > 1$, there are classes whose labeled growth rate oscillates between $n^{(c+o(1))n}$ and $2^{n^{2-1/c}}$ infinitely often, and so there can be no gaps in the possible growth rates in this range. But our result suggests that there may be significant structure within the $n^{(1+o(1))n}$ range.

Finally, in the course of proving our main theorem, we also obtain a new characterization of graph classes of bounded shrub-depth, as arising from certain very tame infinite graphs.

Definition 4. A countable graph G is *oligomorphic* if for every $n \geq 1$, the diagonal action of the automorphism group $\text{Aut}(G)$ on G^n has only finitely many orbits.

We refer to [4] for a definition of what it means for a graph G to be monadically stable, but it roughly means that even after an arbitrary vertex coloring, the graph G cannot encode (in first-order logic) arbitrarily large finite linear orders.

Theorem 5. *A hereditary graph class $\mathcal{C}$ has bounded shrub-depth if and only if there are oligomorphic monadically stable graphs $G_1, \dots, G_k$ such that, letting $\mathcal{D}_i$ be the class of finite induced subgraphs of G_i , we have $\mathcal{C} = \bigcup_{i \in [k]} \mathcal{D}_i$.*

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EVOLUTIONARY PERMUTATIONS PART 2

Miles Jones

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This poster is based on joint work with Oleg Bychenkov, Thiago Donato, Patrick Fong, David Saldanha Da Luz Fontes, Kate Jackson, Nicole Li, Mallory Maddox, Andrew Russell, Luke Wang, Sharon Yang

Imagine a theoretical organism whose genome is not a string of nucleotides but a permutation of length n . Instead of speed or strength or resistance to disease, its viability depends on the frequency of a particular pattern in its genome: organisms with more occurrences of the pattern are more likely to reproduce and survive. Over successive generations, the population evolves toward permutations that optimize for high density of the pattern.

Following this analogy, we continue to study pattern density using evolutionary algorithms. Given a pattern q , recall that $M_{n,q}$ denotes the maximum number of occurrences of pattern q in an n -permutation. We will call the n -permutation that achieves this maximum an *optimizer* and any n -permutation that comes close, we will call a near-optimizer. For some simple patterns, these optimizers are well-known but there are still many patterns out there for which the shape of their optimizer is unknown.

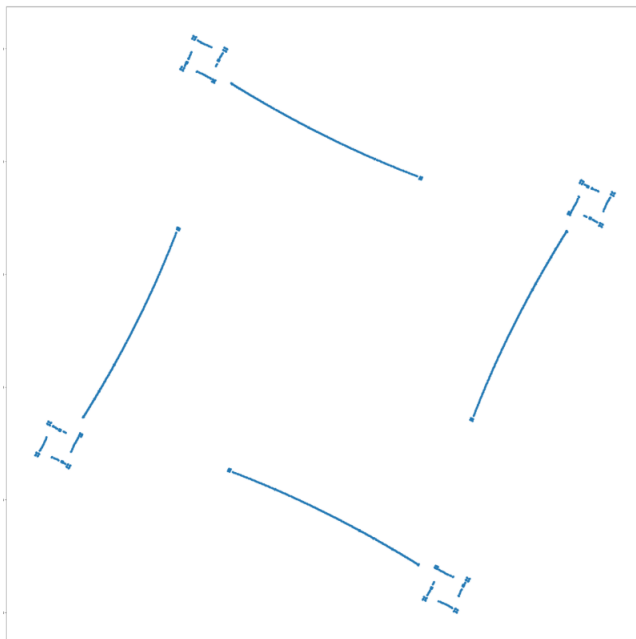


Figure 1: $n = 1024$ near-optimizer with 4,798,426,758 occurrences of 2413

In this poster, we will share the ingredients needed to simulate the evolutionary behavior of permutations (reproduction, mutation, selection) and we will highlight the results of some of our computational experiments, illustrating how these methods can shed light on the general shape of optimizers for patterns and sets of patterns.

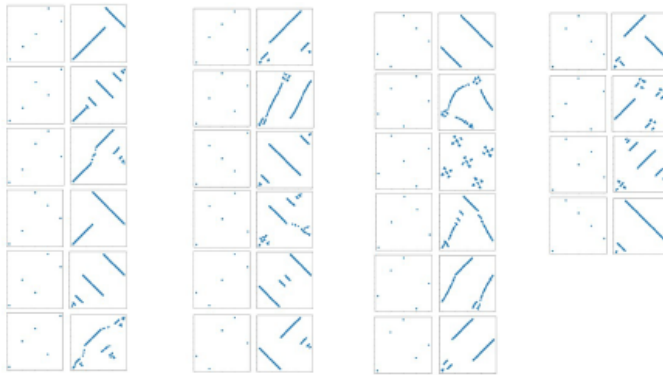


Figure 2: $n = 80$ near-optimizers for all length 5 patterns (up to rotation and reflection symmetries) excluding the increasing pattern.

A few of the experiments we will showcase: we have tried our best to find near-optimizers for the pattern 2413 for permutations up to size $n = 1024$. These results confirm placements of recursion bubbles stated in [4]. We will also share a snapshot of the near-optimizers for each small patterns of length 3, 4, 5 and 6 (up to rotation and reflection symmetries). Some of these have been studied in [1])

A special thanks to Walter Stromquist for some insightful conversations.

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PATTERN AVOIDANCE IN LATTICE AND YAMANOUCHI WORDS

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This poster is based on joint work with Rebecca Smith

This research focuses on pattern avoidance in binary lattice and Yamanouchi words. Pattern avoidance in words has been studied by many including Burstein [1].

We determine the number of such words of size n that avoid a specific pattern in several specific cases. A *lattice word* (sometimes referred to as a ballot sequence) when read from left to right always has at least as many entries of value i as it has of value $i + 1$. These words were noted by Dukes [2] as being ascent sequences. A *Yamanouchi word* is a reversed lattice word.

For example, $\pi = 1213234$ is a lattice word, but $\sigma = 1213321$ is not because the second 3 shows up before the second 2. Similarly, $\tau = 4323121$ is an example of a Yamanouchi word but $\sigma^{rev} = 1233121$ is not.

We study classical pattern avoidance on these sets of words when restricted to the binary case.

We utilize some of the following foundational results about the structure of these words.

Lemma 1. *A lattice word w avoids a pattern $\mu = \mu_1\mu_2 \dots \mu_k$ where $\mu_1 \neq 0$ if and only if w avoids the pattern 0μ .*

Corollary 2. *A Yamanouchi word v avoids a pattern μ where the last entry of μ is nonzero if and only if v avoids the pattern $\mu 0$.*

Observation 3. The number of lattice words of size n that avoid a pattern μ is equal to the number of Yamanouchi words of size n that avoid the reverse of μ .

Our enumerative results include words avoiding patterns of lengths up to four including 010, 101, and 110.

Proposition 4. *The number of binary Yamanouchi words of size n avoiding the pattern 010 is $\lfloor \frac{n}{2} \rfloor + 1$ which is A008619 in the OEIS [3].*

Theorem 5. *The number of binary Yamanouchi words of size n that avoid the pattern 101 is*

$$Y_n = \left\{ \begin{array}{ll} 1, & \text{if } n = 0 \\ Y_{n-1} + \lfloor \frac{n}{2} \rfloor, & \text{if } n \geq 1 \end{array} \right\} = \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

This sequence is A033638 [3].

Theorem 6. *The number of binary Yamanouchi words of size n that avoid the pattern 011 is given by A267528 [3], that is*

$$F_n = \left\{ \begin{array}{ll} 1, & n = 0, 1 \\ n + \lfloor \frac{n-2}{2} \rfloor, & n \geq 2 \end{array} \right.$$

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