

Two Wilf-equivalences related to Chebyshev polynomials

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Permutation Patterns 2026
Cal State LA
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Joint work with Eric Egge and Sergi Elizalde

1 Problem

2 Results and conjectures

3 Total occurrences

4 Questions

Definitions

- Layered permutations, $\text{Av}(312, 231)$

$$[m] = \underbrace{1 \oplus \cdots \oplus 1}_m = m \ m-1 \ m-2 \ \dots \ 2 \ 1$$

$$[m_1, m_2, \dots, m_k] = [m_1] \oplus [m_2] \oplus \cdots \oplus [m_k]$$

- For example,

$$\underbrace{321}_3 \underbrace{7654}_4 \underbrace{98}_2 = 321 \oplus 4321 \oplus 21 = [3] \oplus [4] \oplus [2] = [3, 4, 2]$$

- Consider the following generating functions, on permutations and involutions, respectively:

$$F_{\pi}(x) = \sum_{n=0}^{\infty} |\text{Av}_n(312, \pi)| x^n$$

$$G_{\pi}(x) = \sum_{n=0}^{\infty} |I_n(3412, \pi)| x^n$$

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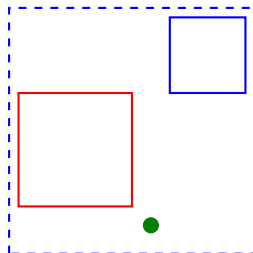
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The recurrence relation (few layers)

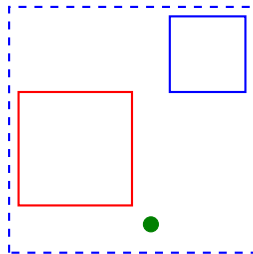


$$F_{[m]} = 1 + x F_{[m-1]} F_{[m]} = \frac{1}{1 - x F_{[m-1]}}$$

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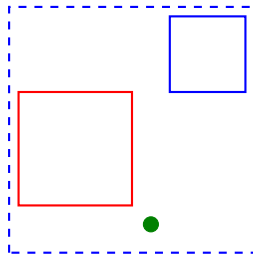


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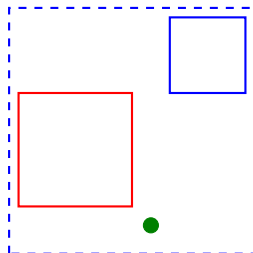


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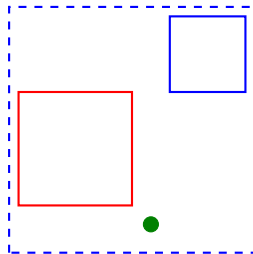


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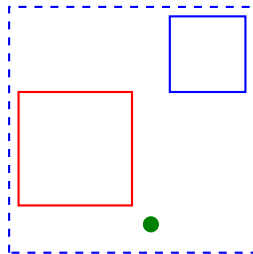
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A diagram showing a square with a dashed blue border. Inside this square, there is a red square on the left and a blue square in the top right corner. A green dot is located at the bottom center of the dashed blue square.

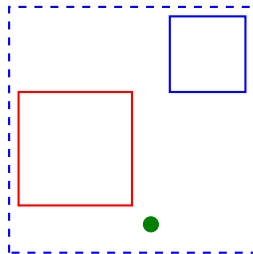


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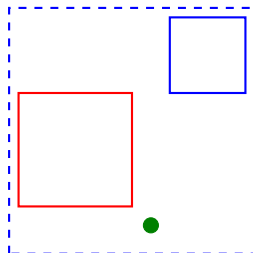


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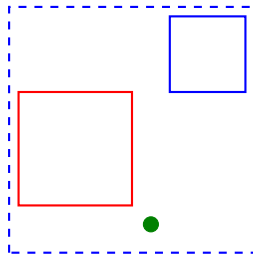
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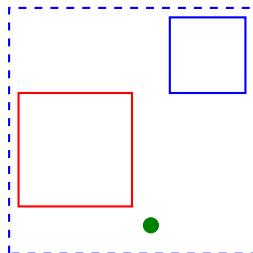
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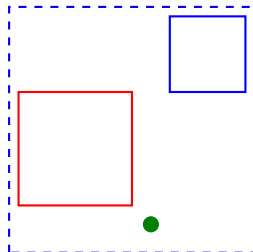


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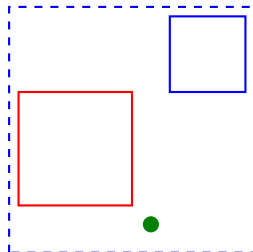


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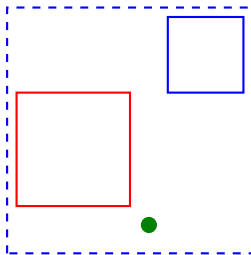


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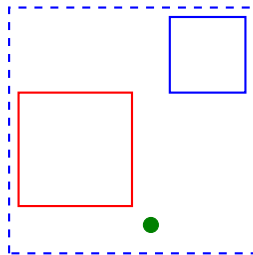
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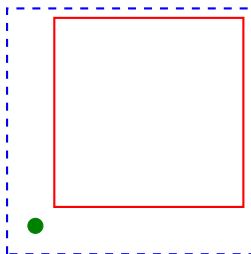
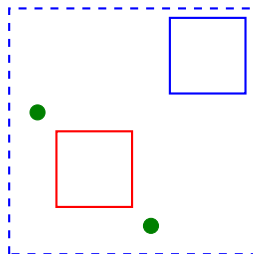
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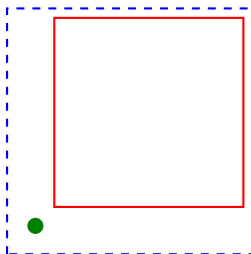
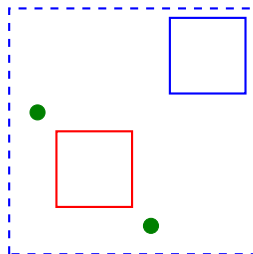
A similar structure on 3412-avoiding involutions (few layers)



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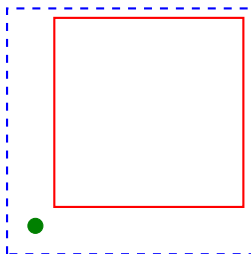
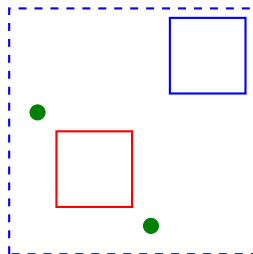
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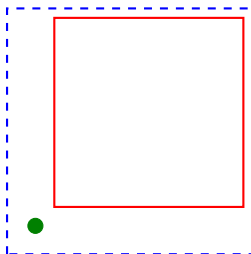
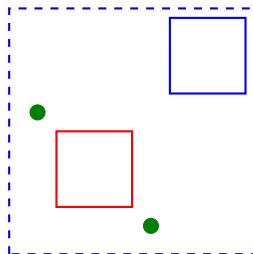
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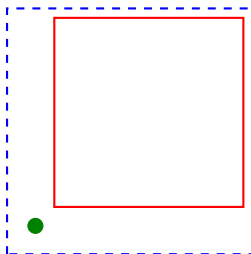
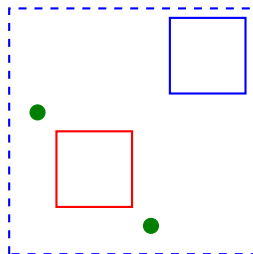
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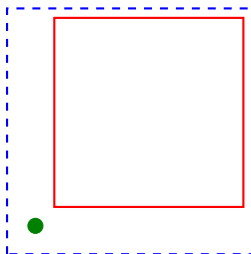
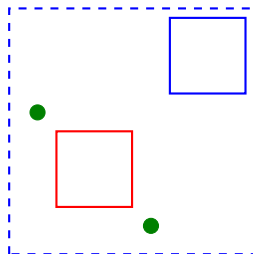
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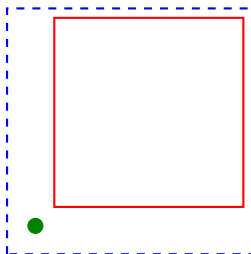
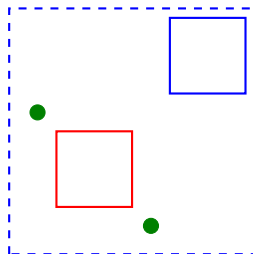


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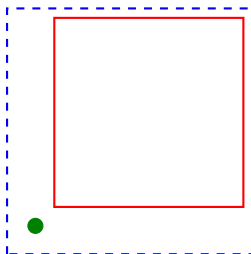
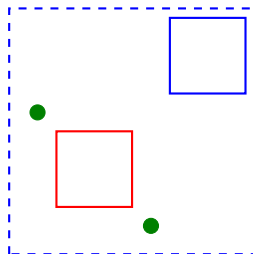


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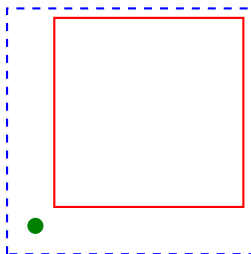
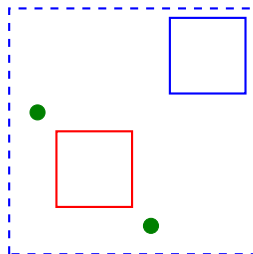
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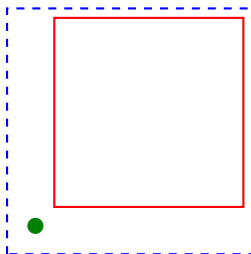
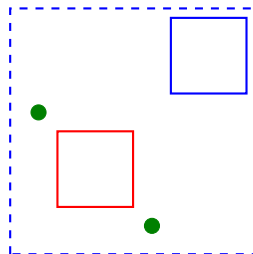


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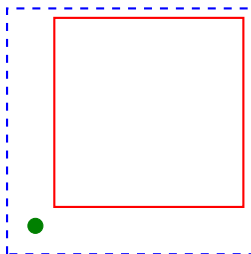
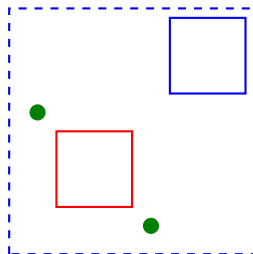
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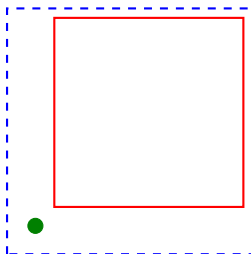
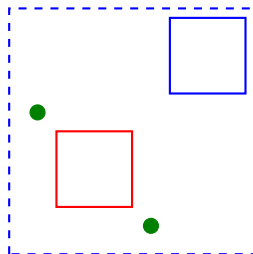
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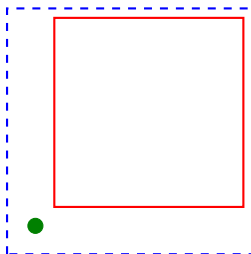
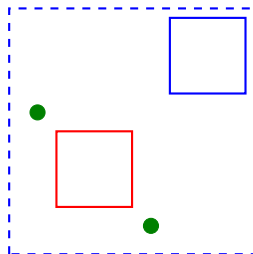
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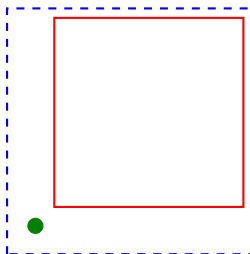
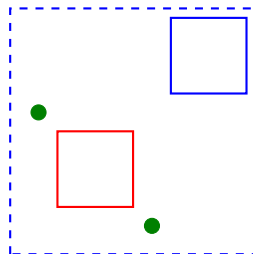


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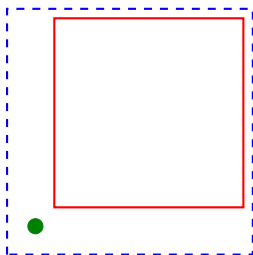
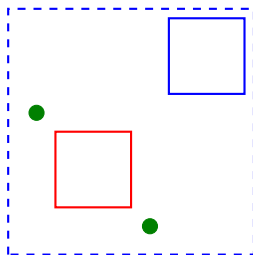


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The recurrence relation (more layers)



$$\begin{aligned}
 G_{[m_1, \dots, m_k]} = & 1 + x G_{[m_1, \dots, m_k]} + x^2 G_{[m_1-2]} G_{[m_1, \dots, m_k]} + x^2 (G_{[m_1, m_2]} - G_{[m_1-2]}) G_{[m_2, \dots, m_k]} \\
 & + \sum_{j=3}^k x^2 (G_{[m_1, \dots, m_j]} - G_{[m_1, \dots, m_{j-1}]}) G_{[m_j, \dots, m_k]}
 \end{aligned}$$

Chebyshev polynomials of the second kind

$$U_n(\cos x) = \frac{\sin((n+1)x)}{\sin x}$$

$$U_{-1}(x) = 0, \quad U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x, \quad \dots$$

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Previous results

Theorem (Mansour-Vainshtein, 2001)

$$F_{[m]}(x) = \frac{2xU_{m-1}}{U_m} \circ \frac{1}{2\sqrt{x}}$$

$$F_{[m_1, m_2]}(x) = \frac{2xU_{m_1+m_2-1}}{U_{m_1+m_2}} \circ \frac{1}{2\sqrt{x}} = F_{[m_1+m_2]}(x)$$

$$F_{[m_1, m_2, m_3]}(x) = \frac{2x \left(U_{m_1+m_2+m_3} U_{m_1+m_2+m_3-1} + U_{m_1+m_2-1} U_{m_1+m_3-1} U_{m_2+m_3-1} \right)}{U_{m_1+m_2} U_{m_1+m_3} U_{m_2+m_3}} \circ \frac{1}{2\sqrt{x}}$$

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Theorem (Egge, 2004)

$$G_{[2m]}(x) = \frac{1}{\sqrt{x}} \cdot \frac{U_{m-1}}{U_m} \left(\frac{1-x}{2\sqrt{x}} \right)$$

$$G_{[2m-1]}(x) = \frac{1}{\sqrt{x}} \cdot \frac{U_{m-1} + U_{m-2}}{U_m + U_{m-1}} \left(\frac{1-x}{2\sqrt{x}} \right)$$

$$G_{[m_1, m_2]}(x) = G_{[m_1 + m_2]}(x)$$

$$G_{[2m_1, 2m_2, 2m_3]}(x) = \frac{1}{\sqrt{x}} \cdot \frac{U_{m_1+m_2+m_3} U_{m_1+m_2+m_3-1} + U_{m_1+m_2-1} U_{m_1+m_3-1} U_{m_2+m_3-1}}{U_{m_1+m_2} U_{m_1+m_3} U_{m_2+m_3}} \left(\frac{1-x}{2\sqrt{x}} \right)$$

$$F_{[m_1, \dots, m_k]} = G_{[2m_1, \dots, 2m_k]} \left(\frac{\sqrt{x}}{1 + \sqrt{x} - x} \right)$$

The 312-conjecture and the 3412-conjecture

Conjecture (Egge, 2004)

$F_{[m_1, \dots, m_k]}$ and $G_{[m_1, \dots, m_k]}$ are invariant under permutations of $(m_1, \dots, m_k)$.

Equivalently, for any $\tau \in S_k$,

- $(312, [m_1, \dots, m_k])$ and $(312, [m_{\tau(1)}, \dots, m_{\tau(k)}])$ are Wilf-equivalent on permutations; and
- $(3412, [m_1, \dots, m_k])$ and $(3412, [m_{\tau(1)}, \dots, m_{\tau(k)}])$ are Wilf-equivalent on involutions.

Checked: True for $k \leq 7$.

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New results

Theorem

Let $\pi_1, \pi_2 \in \text{Av}(312)$. Then

$$F_{(\pi_1 \ominus 21) \oplus (\pi_2 \ominus 1)} = F_{(\pi_1 \ominus 1) \oplus (\pi_2 \ominus 21)} = \frac{1 - xF_{\pi_1 \ominus 1} F_{\pi_2 \ominus 1}}{1 - xF_{\pi_1 \ominus 1} - xF_{\pi_2 \ominus 1}}$$

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$$F_{1 \oplus \pi} = F_{\pi \ominus 1} = \frac{1}{1 - xF_{\pi}}.$$

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Consider the class $\mathcal{M}$ of (312-avoiding) patterns obtained by applying the following operations:

$$(\ominus) \quad \sigma \in \mathcal{M} \mapsto \sigma \ominus 1$$

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with no two consecutive applications of $(\oplus)$.

This corresponds to unary-binary trees (growing from bottom right to top left), e.g.
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Descents and ascents

Let $\text{des} = (\underline{21})$, $\text{asc} = (\underline{12})$ be the descent and ascent statistics.

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The 4 joint distributions of pairs of statistics (des, asc) and (asc, des) on $F_{[\pi_1 \ominus 21, \pi_2 \ominus 1]}$ and $F_{[\pi_1 \ominus 1, \pi_2 \ominus 21]}$ are equal.

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A generalization of the 312- and 3412-conjectures

Conjecture

- Let $\pi \in \text{Av}(312)$. Then $F_{[m_1, \dots, m_k] \oplus (\pi \ominus 1)}$ is invariant under permutations of $(m_1, \dots, m_k)$.
- Let $\pi \in \text{Av}(3412)$. Then $G_{[m_1, \dots, m_k] \oplus (\pi \ominus 1)}$ is invariant under permutations of $(m_1, \dots, m_k)$.

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Total occurrences

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Let $m_1, m_2, \dots, m_k \geq 1$, and let $\sigma \in S_k$. Then

*the total number of occurrences of the pattern $[m_1, m_2, \dots, m_k]$
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*the total number of occurrences of the pattern $[m_{\sigma(1)}, m_{\sigma(2)}, \dots, m_{\sigma(k)}]$
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Tunnels in Dyck paths, stacks of tunnels, and 312-avoiders

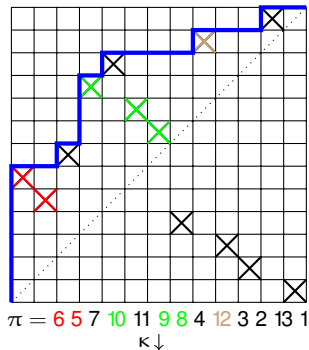
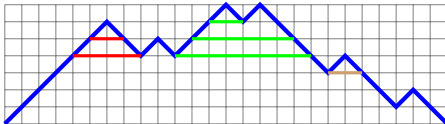


Figure: Krattenthaler's bijection $\text{Av}_n(312) \leftrightarrow \mathcal{D}_n$. A set of nested tunnels is a **stack** of tunnels. A list of non-overlapping stacks from left to right is a **tunnel collection**. An occurrence of the pattern $[2, 3, 1]$ becomes a tunnel collection of type $(2, 3, 1)$, the list of stack sizes from left to right.

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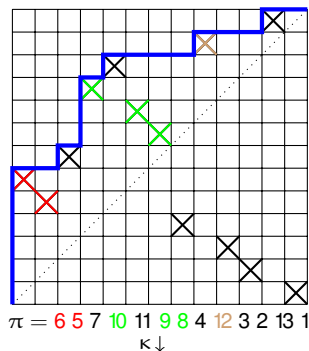
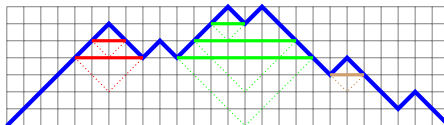


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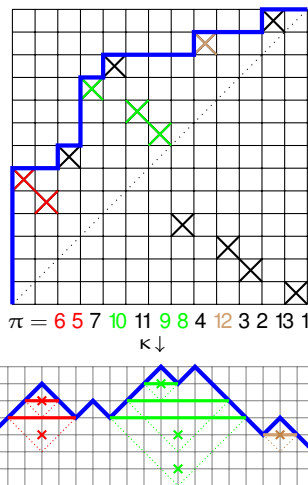


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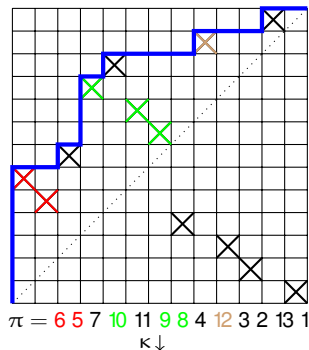
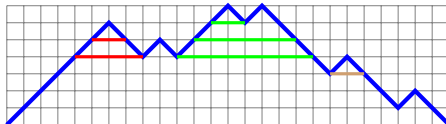


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Bijection: occurrences of layered patterns $\leftrightarrow$ tunnel collections

Lemma

Let $\pi \in \text{Av}_n(312)$ and let $D = \kappa(\pi) \in \mathcal{D}_n$.

A subsequence of π is an occurrence of the pattern $[m_1, m_2, \dots, m_k]$

if and only if

the corresponding tunnels in D form a tunnel collection of type $(m_1, m_2, \dots, m_k)$.

The tunnel collection bijection

Given an occurrence of $[m_1, m_2, \dots, m_k]$ in $\pi \in \text{Av}(312)$, consider the corresponding tunnel collection $(m_1, m_2, \dots, m_k)$ in the corresponding Dyck path $D = \kappa(\pi)$.

Let D_i be the (raised) Dyck subpath corresponding to the bottom tunnel in the i -th stack in our tunnel collection.

This yields the decomposition

$$D = A_0 D_1 A_1 D_2 A_2 \dots A_{k-1} D_k A_k.$$

Then

$$D' = A_0 D_{\sigma(1)} A_1 D_{\sigma(2)} A_2 \dots A_{k-1} D_{\sigma(k)} A_k$$

contains the tunnel collection of the type $(m_{\sigma(1)}, m_{\sigma(2)}, \dots, m_{\sigma(k)})$,

so $\pi' = \kappa^{-1}(D')$ contain an occurrence of pattern $[m_{\sigma(1)}, m_{\sigma(2)}, \dots, m_{\sigma(k)}]$.

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so $\pi' = \kappa^{-1}(D')$ contain an occurrence of pattern $[m_{\sigma(1)}, m_{\sigma(2)}, \dots, m_{\sigma(k)}]$.

The tunnel collection bijection

Given an occurrence of $[m_1, m_2, \dots, m_k]$ in $\pi \in \text{Av}(312)$, consider the corresponding tunnel collection $(m_1, m_2, \dots, m_k)$ in the corresponding Dyck path $D = \kappa(\pi)$.

Let D_i be the (raised) Dyck subpath corresponding to the bottom tunnel in the i -th stack in our tunnel collection.

This yields the decomposition

$$D = A_0 D_1 A_1 D_2 A_2 \dots A_{k-1} D_k A_k.$$

Then

$$D' = A_0 D_{\sigma(1)} A_1 D_{\sigma(2)} A_2 \dots A_{k-1} D_{\sigma(k)} A_k$$

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A tunnel collection bijection illustration

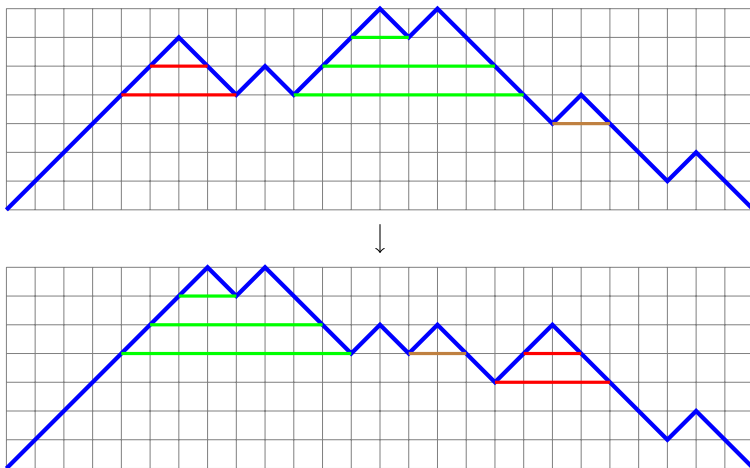


Figure: The pair (D, T) above, where T is a tunnel collection of type $(2, 3, 1)$, is mapped to the pair (D', T') below, where T' is tunnel collection of type $(3, 1, 2)$. Here $\sigma = 231$ and $D = A_0 \textcolor{red}{D}_1 A_1 \textcolor{green}{D}_2 A_2 \textcolor{brown}{D}_3 A_3 \rightarrow A_0 \textcolor{green}{D}_2 A_1 \textcolor{brown}{D}_3 A_2 \textcolor{red}{D}_1 A_3 = D'$.

A generalization

Under this bijection everything above each bottom tunnel is preserved, so it can be generalized as follows.

Theorem

Let $\tau_1, \tau_2, \dots, \tau_k$ be arbitrary permutations of any length (possibly empty), and let $\sigma \in S_k$. Then the total number of occurrences of the pattern

$$(\tau_1 \ominus 1) \oplus (\tau_2 \ominus 1) \oplus \dots \oplus (\tau_k \ominus 1)$$

in all permutations in $\text{Av}_n(312)$ equals the total number of occurrences of the pattern

$$(\tau_{\sigma(1)} \ominus 1) \oplus (\tau_{\sigma(2)} \ominus 1) \oplus \dots \oplus (\tau_{\sigma(k)} \ominus 1)$$

in all permutations in $\text{Av}_n(312)$.

A more general generalization

Let $\delta_r = r \dots 21$ for the descending permutation.

Theorem

Let $a_1, a_2, \dots, a_k \geq 1$ and $b_1, b_2, \dots, b_k \geq 1$ be such that $\sum_i a_i = \sum_i b_i$, let $\tau_1, \tau_2, \dots, \tau_k$ be arbitrary permutations of any length (possibly empty), and let $\sigma \in S_k$. Then the total number of occurrences of the pattern

$$(\tau_1 \ominus \delta_{a_1}) \oplus (\tau_2 \ominus \delta_{a_2}) \oplus \dots \oplus (\tau_k \ominus \delta_{a_k})$$

in all permutations in $\text{Av}_n(312)$ equals the total number of occurrences of the pattern

$$(\tau_{\sigma(1)} \ominus \delta_{b_1}) \oplus (\tau_{\sigma(2)} \ominus \delta_{b_2}) \oplus \dots \oplus (\tau_{\sigma(k)} \ominus \delta_{b_k})$$

in all permutations in $\text{Av}_n(312)$.

Some more general questions

- For a (set of) permutation(s) A , call permutations (involutions, etc.) σ_1, σ_2 **A -Wilf-equivalent** if

$$|Av_n(A, \sigma_1)| = |Av_n(A, \sigma_2)|$$

for all $n \geq 0$.

- What are other interesting cases of A -Wilf-equivalence?
 - $A = \{312, 231, 321\}$ (layers of length 1 or 2)
 - $A = \{213, 231, 321\}$ (identity with top element possibly moved left)
- Given (a set of) pattern(s) A , which (sets of) patterns are both Wilf-equivalent and A -Wilf-equivalent?
 - E.g. 231 and 321 are both Wilf-equivalent and 213-Wilf-equivalent,
 - E.g. 2413 and (2143, 415263) are both Wilf-equivalent (conjecture!) and 3142-Wilf-equivalent.

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