A charge-induced DAG structure on permutations

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Joint work (in progress) with L. Lapointe and M. Weiner

Charge and cocharge vectors

Let $\sigma \in S_n$ and let $i_k = \mathrm{pos}_\sigma(k)$. Set

$$\mathbf{ch}(\sigma)_{i_1} = 0 = \overline{\mathbf{ch}}(\sigma)_{i_1}.$$

For $k = 2, 3, \ldots, n$:

- If $i_k > i_{k-1}$ (value k is to the right of value $k-1$):

$$\mathbf{ch}(\sigma)_{i_k} = \mathbf{ch}(\sigma)_{i_{k-1}} + 1 \quad \text{and} \quad \overline{\mathbf{ch}}(\sigma)_{i_k} = \overline{\mathbf{ch}}(\sigma)_{i_{k-1}}.$$

- If $i_k < i_{k-1}$ (value k is to the left of value $k-1$):

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Example $\sigma = 37158264$

σ	3	7	1	5	8	2	6	4
$\mathbf{ch}(\sigma)$								

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$$\mathbf{ch}(\sigma) = (1,3,0,2,4,1,3,2), \qquad \overline{\mathbf{ch}}(\sigma) = (1,3,0,2,3,0,2,1).$$

$$\mathbf{ch} + \overline{\mathbf{ch}} + \mathbf{1} = \sigma \quad \rightsquigarrow \quad \mathrm{ch}(\sigma) + \overline{\mathrm{ch}}(\sigma) = \binom{n}{2}$$

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In particular,

$$\mathbf{ch}(\mathrm{id}_n) = (0,1,2,\ldots,n-1) \text{ and } \overline{\mathbf{ch}}(\mathrm{id}_n) = (0,0,0,\ldots,0).$$

Main properties

The map $\sigma \mapsto \mathbf{ch}(\sigma)$ is a bijection between S_n and the set of vectors $v = (v_1, v_2, \ldots, v_n) \in \mathbb{N}_0^n$ satisfying:

1. The entries of v form a contiguous interval $\{0, 1, \ldots, M\}$ for some $0 \leq M < n$.
2. For each $1 \leq k \leq M$, the leftmost $k-1$ precedes the rightmost k.

- Interaction with symmetries:

$$\mathbf{ch}(\sigma) = \overline{\mathbf{ch}}(\sigma^r)^r, \qquad \overline{\mathbf{ch}}(\sigma) + \mathbf{ch}(\sigma^c) = M \cdot \mathbf{1},$$

where $M = \max(\overline{\mathbf{ch}}(\sigma)) = \max(\mathbf{ch}(\sigma^c))$.

Other properties (related to pattern avoidance)

- $\max(\mathbf{ch}(\pi \oplus \tau)) = \max(\mathbf{ch}(\pi)) + \max(\mathbf{ch}(\tau)) + 1$.
- $\max(\mathbf{ch}(\pi \ominus \tau)) = \max(\mathbf{ch}(\pi)) + \max(\mathbf{ch}(\tau))$.
- If σ avoids any pattern in $\{132, 213, 231, 312\}$, then $\max(\mathbf{ch}(\sigma)) = \mathrm{asc}(\sigma)$.
- If σ avoids any pattern in $\{123, 321\}$, then $\max(\mathbf{ch}(\sigma)) = \mathrm{asc}(\sigma^{-1})$.

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  $\mathbf{ch}(2413) = (0, 1, 0, 1)$ and $\mathbf{ch}(3142) = (1, 0, 2, 1)$.
- $\sigma \in S_n$ has a nondecreasing charge vector if and only if σ is layered (avoids 231 and 312).

The charge-cocharge relation

For $\sigma, \tau \in S_n$, we write $\sigma \to \tau$ if the entries of the vector

$$w = \overline{\mathbf{ch}}(\sigma) + \mathbf{ch}(\tau)$$

are all distinct and $\max(w) - \min(w) = n - 1$.

w is a shifted permutation of $\{0, 1, \ldots, n-1\}$

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- $\sigma \to \sigma$.
- If $m = \min(w)$, then $nm = \mathrm{ch}(\tau) - \mathrm{ch}(\sigma)$.

$$\sigma \to \tau \;\Rightarrow\; \mathrm{ch}(\tau) \equiv \mathrm{ch}(\sigma) \pmod{n}.$$

Symmetries and cyclage invariance

The relation is compatible with the standard symmetries:

$$\sigma \to \tau \implies \tau^r \to \sigma^r, \quad \tau^c \to \sigma^c, \quad \sigma^{rc} \to \tau^{rc}.$$

If $\sigma \to \tau$, then the permutation $\sigma \circ \tau^{-1}$ is nonseparable.

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Cyclage invariance. If $\varrho = 2\,3 \cdots n\,1$, then

$$\sigma \to \tau \implies \sigma \circ \varrho \to \tau \circ \varrho.$$

Acyclicity

Theorem

The relation $\to$ on S_n has no directed cycles other than loops:

$$\sigma_1 \to \sigma_2 \to \cdots \to \sigma_\ell \to \sigma_1 \quad \Longrightarrow \quad \sigma_1 = \cdots = \sigma_\ell.$$

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Sketch of Proof. Induction on n.

- By cyclage, assume $\sigma_1(n) = 1$.
- Summing the shifts m around the cycle forces every shift to vanish. . . which propagates $\sigma_i(n) = 1$ to all i.
- Delete the 1 to get a chain in S_{n-1}.

Proof for the case $\ell = 3$

Recall: For every edge $\alpha \to \beta$, $\overline{\mathbf{ch}}(\alpha) + \mathbf{ch}(\beta)$ is a permutation of $\{m, \dots, m+n-1\}$, where $nm = \mathrm{ch}(\beta) - \mathrm{ch}(\alpha) \geq 0$.

Let $a, b, c \geq 0$ be the shifts of $\sigma \to \tau$, $\tau \to \pi$ and $\pi \to \sigma$. Then

$$n(a+b+c) = \big(\mathrm{ch}(\tau) - \mathrm{ch}(\sigma)\big) + \big(\mathrm{ch}(\pi) - \mathrm{ch}(\tau)\big) + \big(\mathrm{ch}(\sigma) - \mathrm{ch}(\pi)\big) = 0$$

so $a = b = c = 0$. Each of the three vectors is a permutation of $\{0, 1, \dots, n-1\}$; in particular each has *exactly one* zero entry.

By cyclage invariance we may assume $\sigma(n) = 1$, so

$$\overline{\mathbf{ch}}(\sigma)_n = 0, \qquad \overline{\mathbf{ch}}(\sigma)_j \geq 1 \quad (j < n).$$

Case $\ell = 3$ continued

Since the vector $w = \overline{\mathbf{ch}}(\sigma) + \mathbf{ch}(\tau)$ has a unique zero entry, we must have $\mathbf{ch}(\tau)_n = 0$, and so $\tau(n) = 1$.

Applying the same argument to the edge $\tau \to \pi$ gives $\pi(n) = 1$.

All three permutations have the 1 in position n. Deleting it and standardizing, yields a cycle

$$\sigma' \to \tau' \to \pi' \to \sigma' \quad \text{in } S_{n-1}.$$

By the induction hypothesis $\sigma' = \tau' = \pi'$, and reinserting the 1 in the last position gives $\sigma = \tau = \pi$. $\square$

Interlude: where the question came from

Luc needed, inside the **affine** 0-**Hecke algebra** $H^{(n)}$ (over $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$), a *complete family of orthogonal idempotents*.

His building block attaches to each σ the element

$$\psi_\sigma \;=\; (-1)^{\ell(\sigma)+\ell(w_0)}\, x^{\mathbf{ch}(\sigma)}\, \partial_{w_0}\, x^{\overline{\mathbf{ch}}(\sigma)}, \quad \partial_{w_0} = \tfrac{1}{a_\delta} \textstyle\sum_w (-1)^{\ell(w)} w.$$

The charge and cocharge appear as **exponent vectors** of Laurent monomials. A single number wouldn't do — vectors are needed

Our combinatorics vs. his algebra

The core computation is $\partial_{w_0} x^\mu \partial_{w_0} = \frac{a_\mu}{a_\delta} \partial_{w_0}$, which gives

$$\psi_w \, \psi_\sigma \propto \frac{a_{\overline{\mathbf{ch}}(w)+\mathbf{ch}(\sigma)}}{a_\delta}, \quad \text{and} \quad a_\mu = 0 \iff \mu \text{ has a repeated entry.}$$

$\mathbf{ch} + \overline{\mathbf{ch}} + \mathbf{1} = \sigma$	$\longleftrightarrow$	$\psi_\sigma^2 = \psi_\sigma$ (idempotency)
$\overline{\mathbf{ch}}(w) + \mathbf{ch}(\sigma)$ distinct	$\longleftrightarrow$	$\psi_w \psi_\sigma \neq 0 \quad (w \to \sigma)$
acyclicity of the arrow graph	$\longleftrightarrow$	chain products vanish

Acyclicity makes the sum

$$L_\sigma = \sum_{\text{chains}} (-1)^\ell \, \psi_\sigma \psi_{w^{(1)}} \cdots \psi_{w^{(\ell)}} \quad \textit{finite}, \text{ and}$$

$\{L_w\}$ is a complete set of orthogonal idempotents with $\sum_w L_w = 1$.

Charge classes and the DAG

Total charge partitions S_n into n residue classes mod n, each of size $(n-1)!$.

Within each residue class, $\sigma \to \tau$ ($\sigma \neq \tau$) defines a **directed acyclic graph** on $(n-1)!$ vertices.

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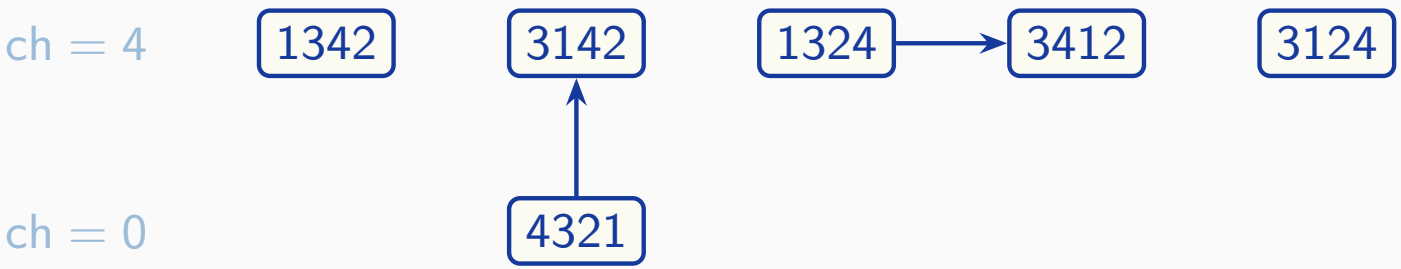
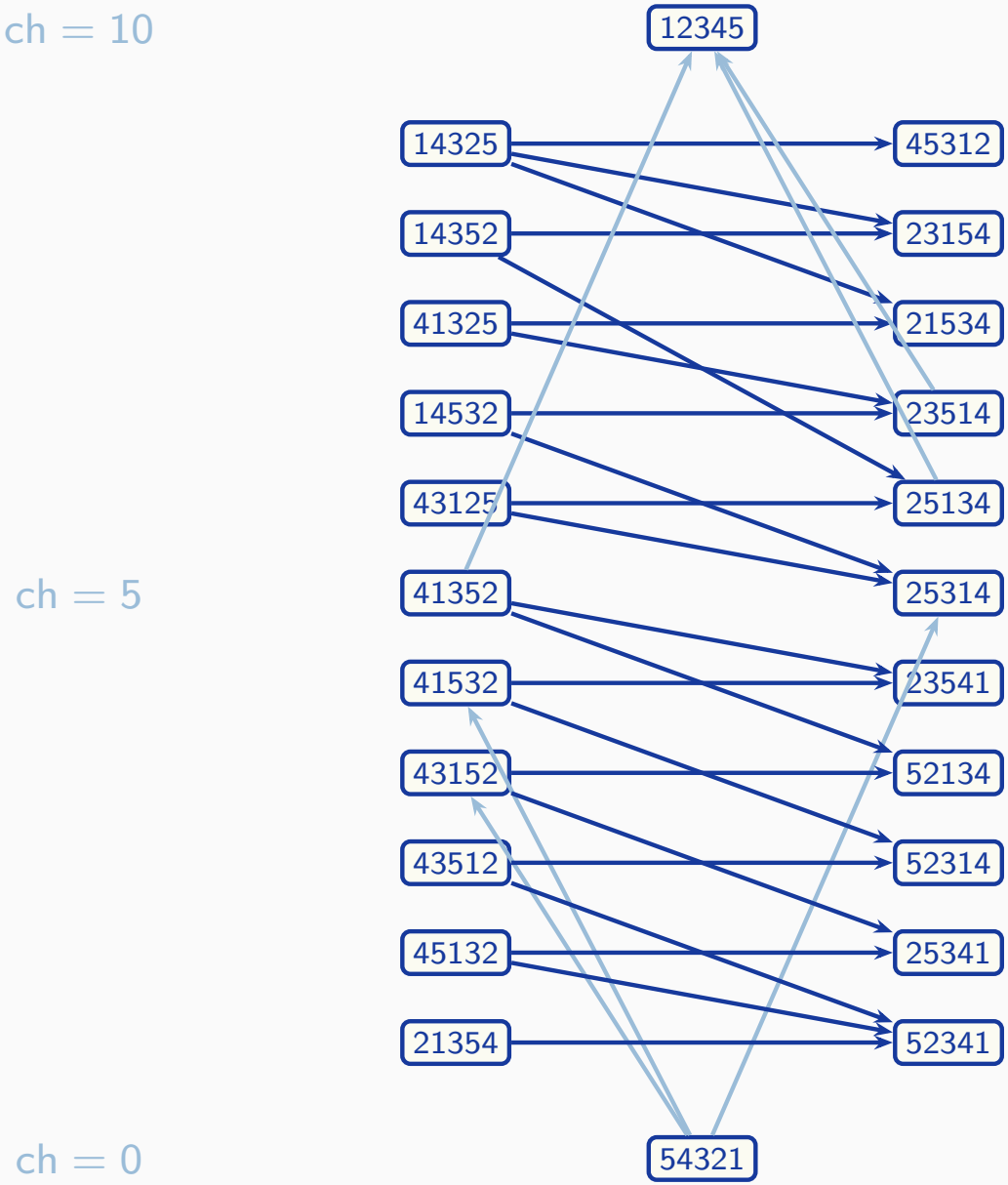
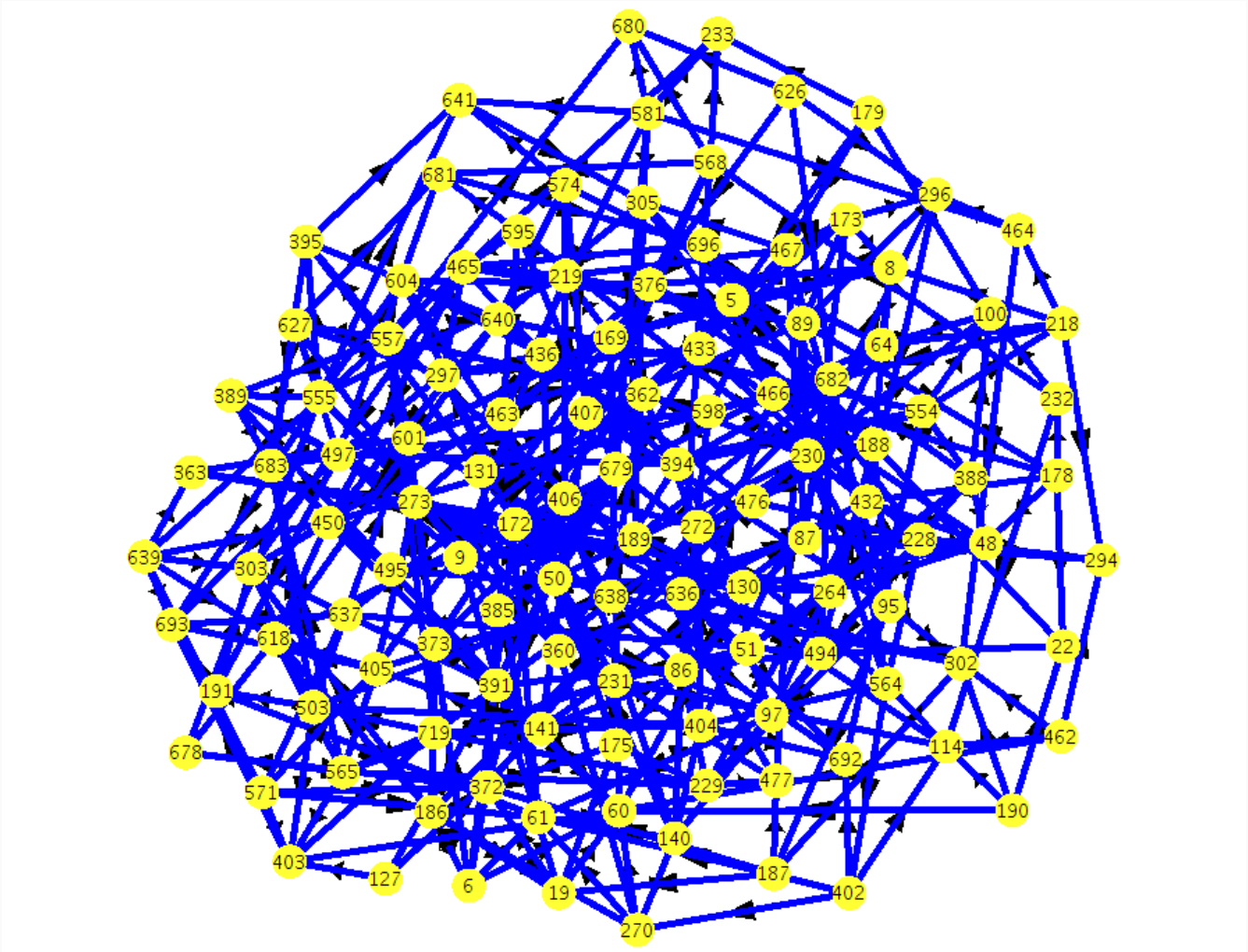
Example: $n = 4$, class ch $\equiv 0 \pmod 4$



Graph for $n = 5$, $\mathrm{ch} \equiv 0 \pmod 5$



Graph for $n = 6$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A new poset $\mathcal{P} = (S_n, \preceq)$

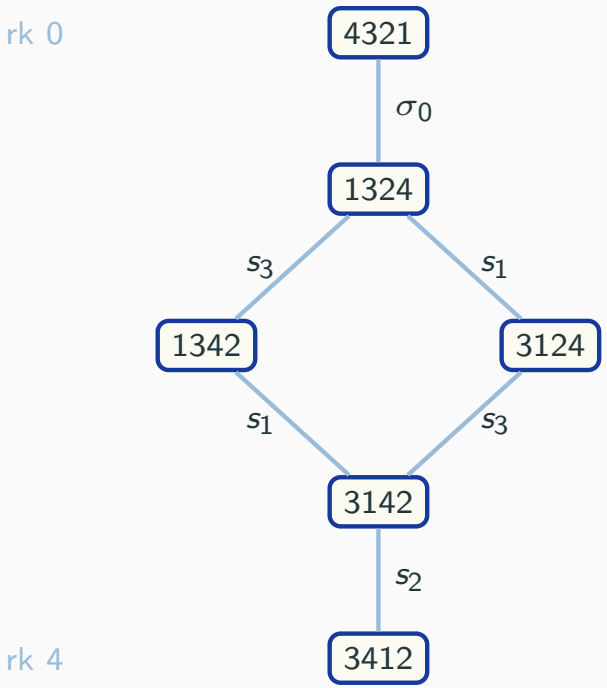
Call i a *long ascent* of w if $w(i+1) > w(i)+1$.

Covering relations: $w \gtrdot w\, s_i$ at each long ascent i $(1 \le i \le n-1)$, together with the **affine cover** $w \gtrdot \sigma_0\, w$ when $w(1) > w(n)+1$, where σ_0 swaps the two end positions.

$\mathcal{P} = (S_n, \preceq)$ is **ranked** with exactly n connected components: the level sets of total charge mod n. Each has $(n-1)!$ elements and a **unique minimum and maximum**.

New structures ($n = 4$, $\mathrm{ch} \equiv 0$)

the poset $\mathcal{P}$



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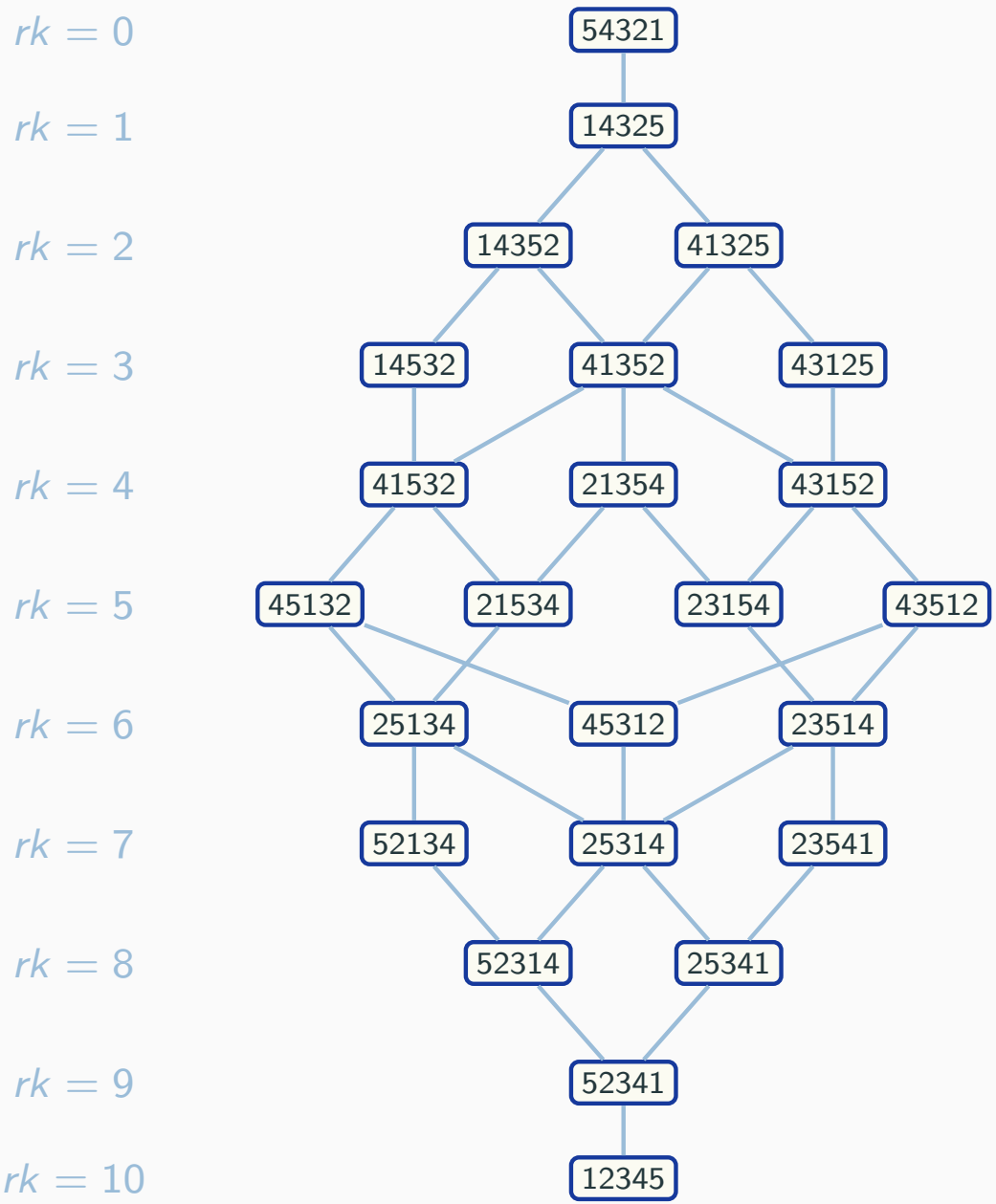

Why $\mathcal{P}$ is the right object

Read one-line notations *cyclically*: deleting $1, \ldots, k$ from w leaves a cyclic word w_k on $\{k+1, \ldots, m\}$. Then

$$rk(w) = \sum_{k=1}^{m-2} \mathrm{ch}\left(w_k\right), \qquad rk(w_0) = 0.$$

(The rank is *not* the total charge!)

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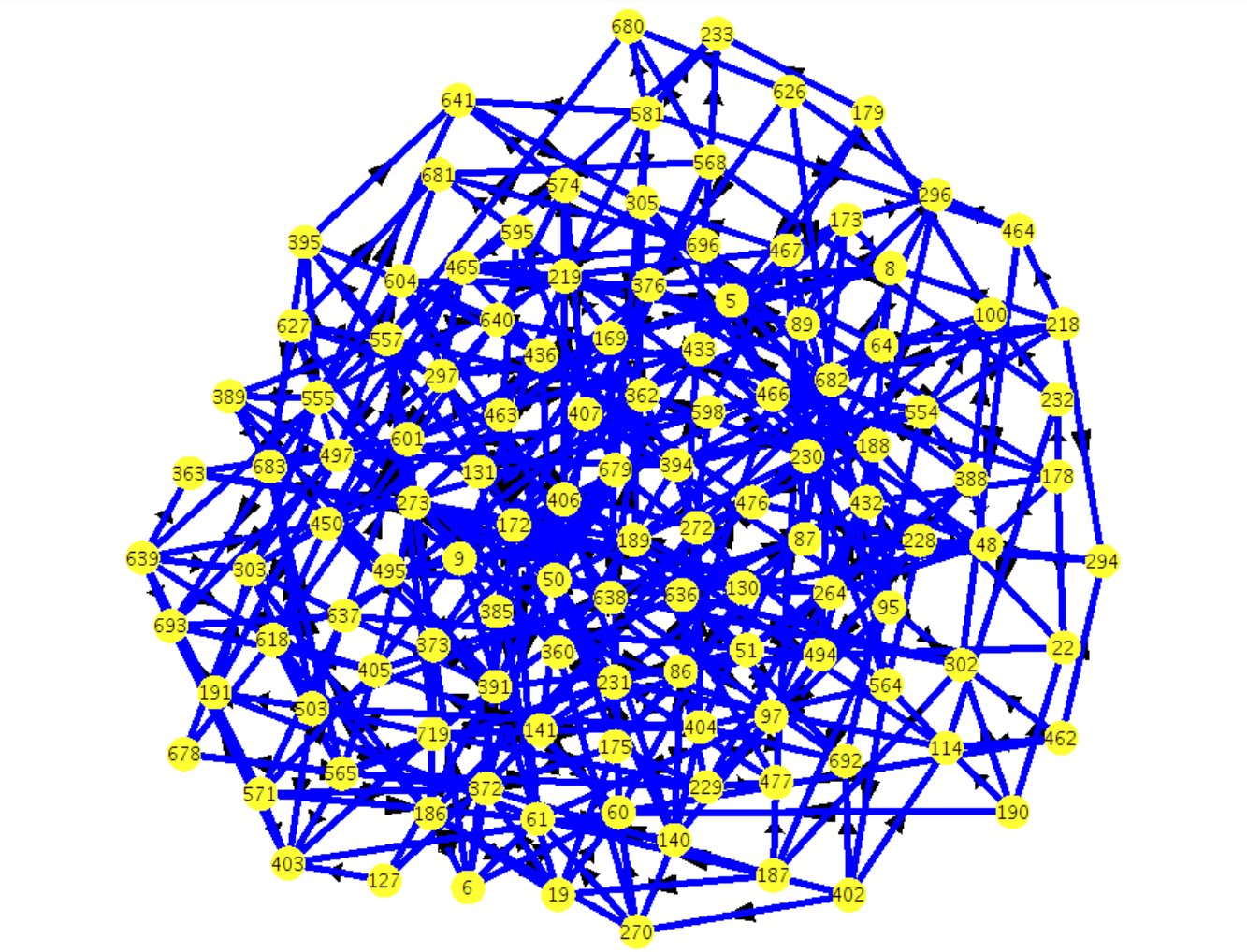
Conjecture: The rank strictly **increases along every arrow** of the charge–cocharge graph.

Graph for $n = 6$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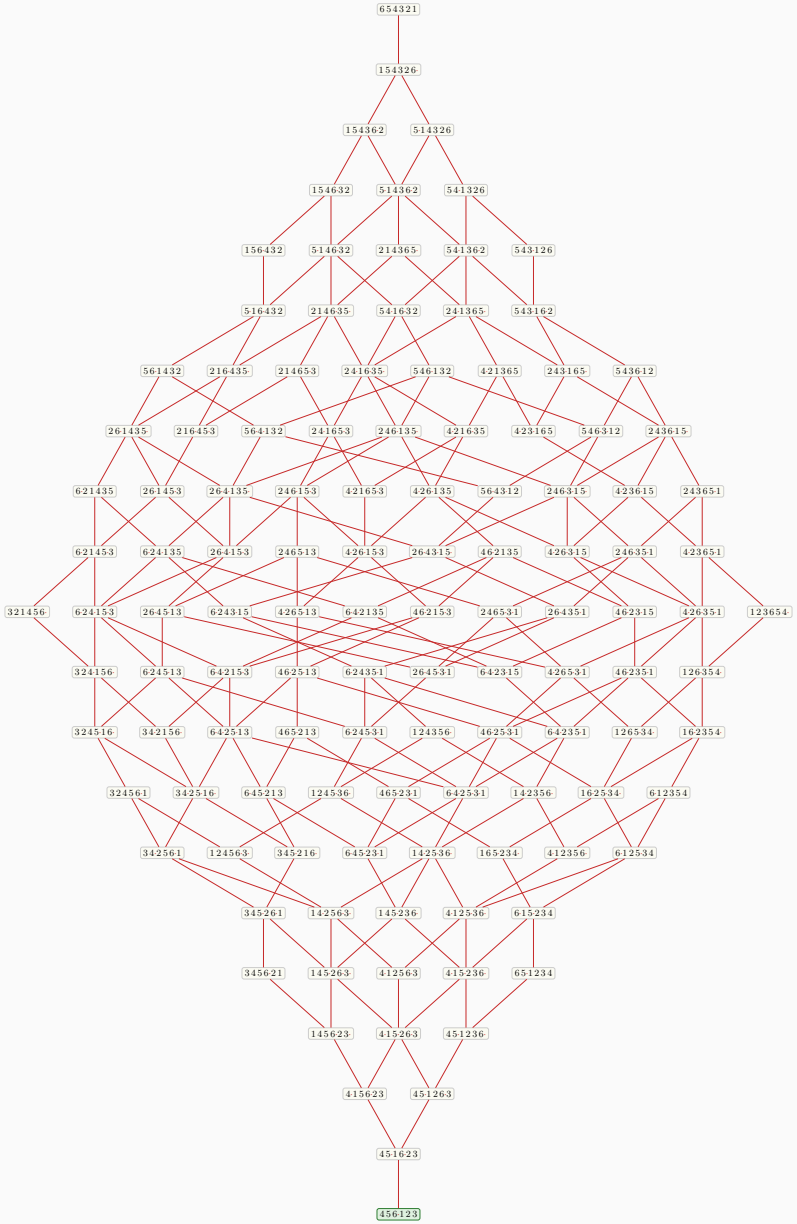
Charge-cocharge arrows for $n = 6$



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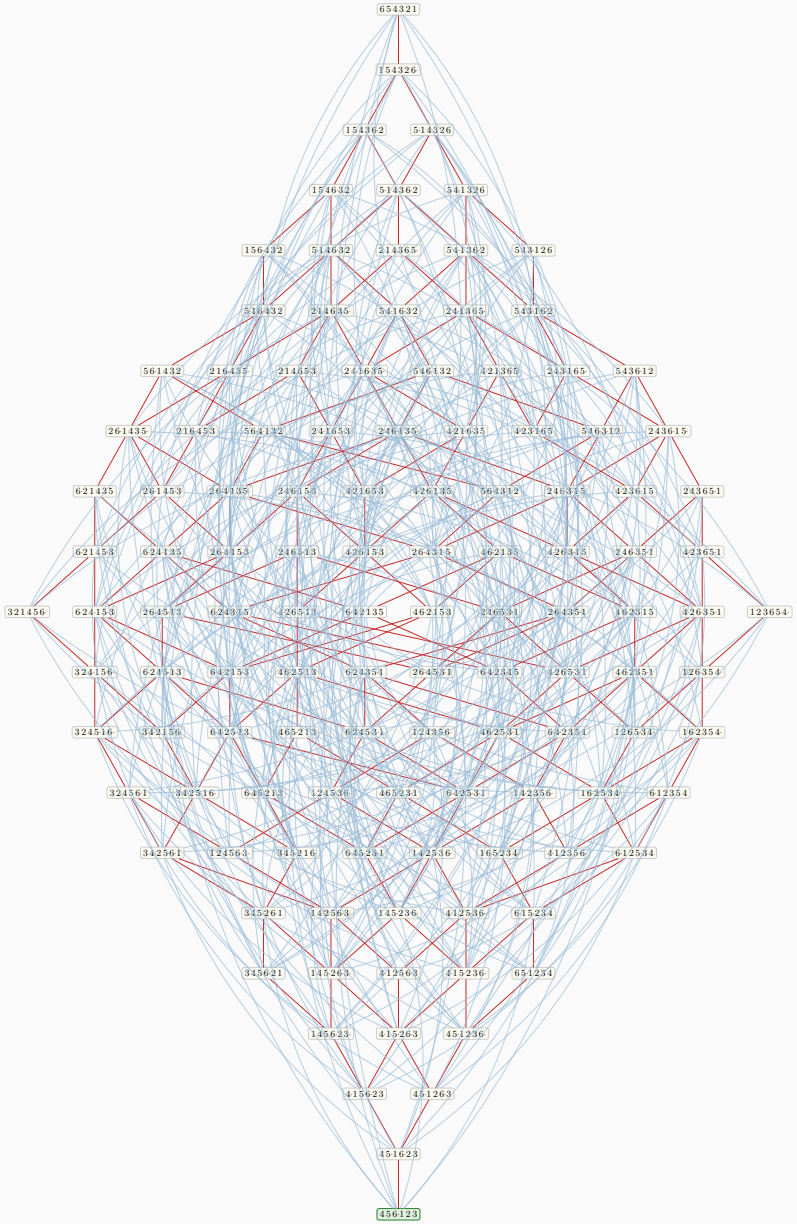

References

- A. Lascoux, M.-P. Schützenberger, *Sur une conjecture de H. O. Foulkes*, C. R. Acad. Sci. Paris **286** (1978), A323–A324.
- L. Lapointe, *A complete family of idempotents in the affine* 0*-Hecke algebra*, preprint.

Thank you!