

On the Schur-positivity of set partitions with a given number of blocks

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Symmetric functions

Definition

- Let $\mathbf{x} = \{x_1, x_2, \dots\}$ be a countably infinite set of commuting variables and let $\mathbb{Q}[[\mathbf{x}]]$ be the algebra of formal power series over $\mathbb{Q}$.
- A power series $f \in \mathbb{Q}[[\mathbf{x}]]$ is called **symmetric** if it has a bounded degree and it is invariant under **any** permutation of variables.
- Equivalently, $f(x_1, x_2, \dots)$ is **symmetric** if for every composition $\alpha = (\alpha_1, \dots, \alpha_k)$, all monomials $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ in f with distinct indices have the same coefficient.

Example

$$f = \sum_{i \neq j} x_i^3 x_j = x_1^3 x_2 + x_1 x_2^3 + x_1^3 x_3 + x_1 x_3^3 + x_2^3 x_3 + x_2 x_3^3 + \dots$$

Quasi-symmetric functions

Definition

A formal power series $f(x_1, x_2, \dots)$ is **quasi-symmetric** if for every composition $(\alpha_1, \dots, \alpha_k)$, all monomials $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ in f with indices $i_1 < i_2 < \cdots < i_k$ have the same coefficients.

Example

$$f = \sum_{i < j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \dots$$

is quasi-symmetric but not symmetric.

Denote by Sym_n (QSym_n) the vector space of (quasi) symmetric functions which are homogeneous of degree n , respectively.

The fundamental basis

For each subset $J \subseteq [n-1]$ define the corresponding **fundamental quasi-symmetric function**

$$F_{n,J}(\mathbf{x}) := \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in J}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Example (for 3 variables)

$$D = \{1\}: F_{3,\{1\}} = x_1 x_2 x_3 + x_1 x_2 x_2 + x_1 x_3 x_3 + x_2 x_3 x_3.$$

$$D = \{2\}: F_{3,\{2\}} = x_1 x_1 x_2 + x_1 x_1 x_3 + x_1 x_2 x_3 + x_2 x_2 x_3.$$

Proposition (Gessel)

The set $\{F_{n,J} \mid J \subseteq [n-1]\}$ is a basis for QSym_n .

Schur-positivity

- The most important basis for Sym_n is the set of Schur functions $\{s_\lambda \mid \lambda \vdash n\}$.
- A symmetric function which can be expanded as a linear combination of Schur functions with positive coefficients is called **Schur positive**.
- Let $\mathcal{B}$ be a (multi-)set of combinatorial objects, equipped with a **descent map** $\text{Des} : \mathcal{B} \rightarrow P([n-1])$, which associates to each element $b \in \mathcal{B}$ a subset $\text{Des}(b) \subseteq [n-1]$.
- Define the quasi-symmetric function:

$$Q_n(\mathcal{B}) := \sum_{b \in \mathcal{B}} m(b, \mathcal{B}) F_{n, \text{Des}(b)},$$

where $m(b, \mathcal{B})$ is the multiplicity of the element $b \in \mathcal{B}$.

Question

- When is $Q_n(\mathcal{B})$ **symmetric** for all n ?
- In that case, when is $Q_n(\mathcal{B})$ **Schur-positive** for all n ?

Schur-positivity in Standard Young tableaux

The **descent set of a standard Young tableau** T is:

$$\text{Des}(T) = \{i \mid i+1 \text{ is in a lower row than } i \text{ in } T\}.$$

Example

Given $T =$

1	3	5
2	4	7
6		

, we have: $\text{Des}(T) = \{1, 3, 5\}$.

Theorem (Gessel)

For every partition $\lambda \vdash n$, $\mathcal{Q}_n(\text{SYT}(\lambda)) = \sum_{T \in \text{SYT}(\lambda)} F_{n, \text{Des}(T)} = s_\lambda$.

Hence, proving Schur-positivity of a set E with respect to some descent function, amounts to defining a descent-preserving function $\varphi : E \rightarrow \bigcup_{\lambda \vdash n} \text{SYT}(\lambda)$, such that for each $\lambda \vdash n$ and $S, T \in \text{SYT}(\lambda)$, one has $|\varphi^{-1}(S)| = |\varphi^{-1}(T)|$.

Some known results on Schur-positivity of set partitions

Definition (Marmor 2024)

We say that a set $J \subset [n-1]$ is **sparse** if $\{j, j+1\} \not\subseteq J$ for every $1 \leq j \leq n-2$.

For a set $\mathcal{A}$ and a function $D : \mathcal{A} \rightarrow 2^{[n-1]}$, we say that D is **sparse** if $D(a)$ is sparse for all $a \in \mathcal{A}$.

Remark

*The property of a sparse set was called **lacunar** by Grinberg (2018)*

Theorem (Marmor 2024)

Let $\mathcal{A}$ be a finite set with a statistic $D : \mathcal{A} \rightarrow 2^{[n-1]}$. Then the following statements are equivalent:

- D is sparse, and for every sparse $J \subseteq [n-1]$, the cardinality of the set

$$\{a \in \mathcal{A} \mid D(a) \supseteq J\}$$

depends on the size of J only.

- $\mathcal{A}$ is symmetric with respect to D , with a Schur expansion of the form $\mathcal{Q}_D(\mathcal{A}) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_k s_{n-k,k}$ for some $c_k \in \mathbb{Z}$.

Furthermore, if these statements hold, then $\mathcal{A}$ is Schur-positive, and its Schur expansion is

$$\mathcal{Q}_D(\mathcal{A}) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} |\{a \in \mathcal{A} \mid D(a) = \{1, 3, 5, \dots, 2k-1\}\}| \cdot s_{n-k,k}.$$

Definition

A **matching** m on a finite set of vertices S is a set partition of S into blocks, each block of size 1 or 2.

A **perfect matching** is a matching without singletons.

Let $m \in \mathcal{M}_n$ be a matching on $[n]$. If $(i, i+1) \in m$, then it is called a **short chord**. Denote:

$$\text{Short}(m) := \{i \in [n-1] \mid (i, i+1) \in m\}.$$

Theorem (Marmor 2024)

Let $n, f \in \mathbb{N}$ be nonnegative integers, and denote $N = 2n + f$. Then the set of matchings of $[N]$ with exactly f singletons is Schur-positive with respect to the statistic short .

Theorem (Yan, Yang, Zhou, 2024)

Perfect matchings are Schur-positive with respect to some non-sparse statistic Des.

Theorem (Bagno-G-Mansour-Safadi, 2026)

- $\mathcal{Q}_n(\mathcal{R}_n(122))$ is **Schur-positive** with respect to ordinary descents of RG-words. Explicitly:

$$\mathcal{Q}_n(\mathcal{R}_n(122)) = \sum_{k=0}^n s_{\lambda_k},$$

where $\lambda_k = (n - k, 1^k)$ is the hook diagram.

- $\mathcal{Q}_n(\mathcal{R}_n^B(01))$ is **Schur-positive** with respect to ordinary descents of RG-words. Explicitly:

$$\mathcal{Q}_n(\mathcal{R}_n^B(01)) = 2s_{(n)} + 2s_{(n-1,1)} + s_{(n-2,1,1)}.$$

The parameter Desing

Definition

- For $\pi \in \mathcal{Set}(n)$, let $\text{Sing}(\pi)$ be the set of singletons of π .
- An element i is a **desing**, if $i \in \text{Sing}(\pi)$ but $i + 1 \notin \text{Sing}(\pi)$.
- $\text{Desing}(\pi) = \{i \mid i \text{ is desing in } \pi\}$.

Example

$\pi = \{\{1\}, \{2, 3\}, \{4\}, \{5\}, \{6, 7, 8\}\}$:

$$\text{Sing}(\pi) = \{1, 4, 5\}$$

$$\text{Desing}(\pi) = \{1, 5\}$$

Main results of Schur-positivity in set partitions (BG 2026)

$$\sum_{\pi \in \mathcal{A}} \mathcal{F}_{n, \text{Desing}(\pi)} = \sum_k \mathcal{C}_{n,k} s_{(n-k,k)}$$

The set $\mathcal{A}$	The coefficients $\mathcal{C}_{n,k}$
$\mathcal{PSet}(n)$	$e_{n-k,k} = \sum_{i=k}^{n-k} \text{Bell}_{\geq 2}(i)$
$\mathcal{NC}(n)$ Non-crossing partitions	$\sum_{i=k}^{n-k} r_i$ r_i are Riordan numbers , which count non-crossing partitions without singletons
$\mathcal{PSet}_S(n)$ Partitions whose block sizes are in set S	$\sum_{t=k}^{n-k} \sum_{\substack{(a_S)_{S \in S} \\ \sum_{S \in S} s a_S = t}} \prod_{S \in S} \frac{t!}{((s!)^{a_S} a_S!)}$
$\mathcal{NC}_S(n)$ Non-crossing partitions whose block sizes are in set S	$\sum_{t=k}^{n-k} \sum_{\substack{(a_S)_{S \in S} \\ \sum_{S \in S} s a_S = t}} \frac{t!}{\left(t + 1 - \sum_{S \in S} a_S\right)! \prod_{S \in S} (a_S!)}$ Based on Kreweras' formula counting non-crossing partitions without singletons whose block sizes are in S
$\mathcal{PSet}(n, b)$ Partitions having b blocks	$e_{n-k,k}^{(b)} = \sum_{t=k}^{n-k} S_{\geq 2}(n-t, b-t)$ $S_{\geq 2}(n,b)$ are the associated Stirling numbers of the second kind or the Ward numbers .

Idea of proof

For a given set partition $\pi \in \mathcal{PSet}(n)$, write

$$\text{Desing}(\pi) = \{d_1 < \cdots < d_k\} \subseteq \{a_1 < \cdots < a_s\} = \text{Sing}(\pi).$$

Define a mapping $\varphi : \mathcal{PSet}(n) \longrightarrow \bigcup_{\substack{\lambda = (n-k, k) \\ 0 \leq k \leq \lfloor \frac{n}{2} \rfloor}} \text{SYT}(\lambda).$

Ex.: $\pi = \{\{\textcolor{red}{1}\}, \{\textcolor{blue}{2}, 3\}, \{4\}, \{\textcolor{red}{5}\}, \{\textcolor{blue}{6}, 7, 8\}\} \longrightarrow T =$

1	3	4	5	8
2	6	7		

.

Then we show that for $T \in \text{SYT}(n-k, k)$, we have:

$$|\varphi^{-1}(\{T\})| = \text{Bell}_{\geq 2}(k) + \cdots + \text{Bell}_{\geq 2}(n-k).$$

Remark: Similar results can be obtained for signed set partitions and colored set partitions.

Visual perspectives - numerical values of $e_{n-k,k}$

n	$e_{n,0}$ [$c_{n,0}$]	$e_{n-1,1}$ [$c_{n-1,1}$]	$e_{n-2,2}$ [$c_{n-2,2}$]	$e_{n-3,3}$ [$c_{n-3,3}$]	$e_{n-4,4}$ [$c_{n-4,4}$]	$e_{n-5,5}$ [$c_{n-5,5}$]
0	1 [1]					
1	1 [1]					
2	2 [1]	0 [1]				
3	3 [1]	1 [2]				
4	7 [1]	2 [3]	1 [2]			
5	18 [1]	6 [4]	2 [5]			
6	59 [1]	17 [5]	6 [9]	1 [5]		
7	221 [1]	58 [6]	17 [14]	5 [14]		
8	936 [1]	220 [7]	58 [20]	16 [28]	4 [14]	
9	4361 [1]	935 [8]	220 [27]	57 [48]	15 [42]	
10	22083 [1]	4360 [9]	935 [35]	219 [75]	56 [90]	11 [42]
11	120336 [1]	22082 [10]	4360 [44]	934 [110]	218 [165]	52 [132]

- In squared brackets are **Catalan's triangle numbers** $c_{n-k,k}$, counting the number of SYT of a given shape (also called **ballot numbers**, which satisfy the recursion

$$c_{n,k} = c_{n-1,k} + c_{n,k-1}, \quad c_{0,0} = 1. \quad \sum_{k=0}^n c_{n,k} = C(n+1).$$

- The n th element of first column $\{e_{n,0}\}$ counts the number of partitions of *up to* n elements without singletons.

Visual perspectives - numerical values of $e_{n-k,k}$ (cont.)

n	$e_{n,0}$ [$c_{n,0}$]	$e_{n-1,1}$ [$c_{n-1,1}$]	$e_{n-2,2}$ [$c_{n-2,2}$]	$e_{n-3,3}$ [$c_{n-3,3}$]	$e_{n-4,4}$ [$c_{n-4,4}$]	$e_{n-5,5}$ [$c_{n-5,5}$]
0	1 [1]					
1	1 [1]					
2	2 [1]	0 [1]				
3	3 [1]	1 [2]				
4	7 [1]	2 [3]	1 [2]			
5	18 [1]	6 [4]	2 [5]			
6	59 [1]	17 [5]	6 [9]	1 [5]		
7	221 [1]	58 [6]	17 [14]	5 [14]		
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- The squared elements $\{e_{n,n}\} = \{\text{Bell}_{\geq 2}(n)\}$ count the set partitions of $[n]$ without singletons.
- A nice visual way to compute $e_{n-k,k}$: start from $e_{n,n-k}$ and go diagonally right and down until reaching a squared number. For each column you come across, sum up the squared number appearing in the top of that column (including first and last).

- New way to present Bell numbers:
$$\text{Bell}(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} e_{n-k,k} c_{n-k,k}.$$

Visual perspectives - numerical values of $e_{n-k,k}^{(b)}$

$b \backslash n$	1	2	3			4				5				
	$k=0$	$k=0$ $k=1$	$k=0$ $k=1$ $k=2$				$k=0$ $k=1$ $k=2$ $k=3$					$k=0$ $k=1$ $k=2$ $k=3$ $k=4$		
2	1	1												
3	1	1	1	1										
4	1	4	1	1	1		1							
5	1	11	1	4	4	1	1	1	1			1		
6	1	26	1	26	11	1	4	4	4	1		1	1	1
7	1	57	1	131	26	1	26	26	11	1		4	4	4
8	1	120	1	547	57	1	236	131	26	1		26	26	26
9	1	247	1	2038	120	1	1807	547	57	1		236	236	131
10	1	502	1	7072	247	1	11488	2038	120	1		2752	1807	547
11	1	1013	1	23437	502	1	64052	7072	247	1		28813	11488	2038

Claim: If $k \leq n-b$ or $k > 2(n-b)$, then: $e_{n-k,k}^{(b)} = e_{(n+1)-(k+1),k+1}^{(b+1)}$.

Visual perspectives - numerical values of $e_{n,0}^{(b)}$

$n \backslash b$	1	2	3	4	5	6	7	8	9	10	11
2	1	1									
3	1	1	1								
4	1	4	1	1							
5	1	11	4	1	1						
6	1	26	26	4	1	1					
7	1	57	131	26	4	1	1				
8	1	120	547	236	26	4	1	1			
9	1	247	2038	1807	236	26	4	1	1		
10	1	502	7072	11488	2752	236	26	4	1	1	
11	1	1013	23437	64052	28813	2752	236	26	4	1	1

- The squared elements (the sequence $\{e_{2b,0}^{(b)}\}$) are known to count the **total partitions** on n .
- For $n \leq 2b$, we have that: $e_{n+1,0}^{(b+1)} = e_{n,0}^{(b)}$.
- $e_{n,0}^{(2)} = A(n-1, 1)$, where $A(n-1, 1)$ is the Eulerian number, which counts $\pi \in S_{n-1}$ with exactly one descent in their one-line notation.
- Some more connections to second-order Eulerian numbers.

Thank you for your attention!!