

The mex statistic on combinatorial structures

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Choose your favourite combinatorial structure.

Typically, its objects can be decomposed into pieces of different weights.

You might be interested in counting objects whose decomposition does not contain pieces of the smallest possible weight. This often gives famous integer sequences:

- ▶ integer compositions without parts equal to 1 $\rightsquigarrow$ (shifted) Fibonacci numbers;
- ▶ Dyck paths without hills $\rightsquigarrow$ Fine numbers [Deutsch and Shapiro, 2001];
- ▶ planar trees without nodes having only one child $\rightsquigarrow$ Riordan numbers [Bernhart, 1999].

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We generalize this situation by introducing a new statistic.

From combinatorial game theory...

$$S \subseteq \mathbb{N}$$

$$\text{mex } S = \min\{n \in \mathbb{N} \mid n \notin S\}$$

Classical notion in combinatorial game theory (Sprague-Grundy theorem).

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Is there a mex for arbitrary combinatorial objects?

- ▶ Define a suitable notion of combinatorial structure.
- ▶ Introduce the mex statistic for an arbitrary (weighted) combinatorial structure.
- ▶ Describe a general method to compute the mex statistic.
- ▶ Show several interesting examples!

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Weighted combinatorial structures

A combinatorial structure $\mathcal{S}$ is determined by

- ▶ the set of its *pieces* $E(\mathcal{S})$;
- ▶ the set of its *instances* $I(\mathcal{S})$ (obtained from $E(\mathcal{S})$ using standard set-theoretic operations, such as union, intersection, cartesian product, powerset, etc.);
- ▶ the set of its *properties* $P(\mathcal{S})$ (conditions that instances are required to satisfy).

$$\mathcal{S} = \{X \in I(\mathcal{S}) \mid X \text{ satisfies } P(\mathcal{S})\}$$

Weight: $w : E(\mathcal{S}) \rightarrow \mathbf{N}$

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$(\mathcal{S}, w)$ is a *weighted combinatorial structure*.

The mex function of (S, w)

For $X \in S$,

$$\text{mex}(X) = \min_{e \vdash X} \{k \in \mathbf{N} \mid w(e) \neq k\}$$

($e \vdash X$ means “ e is a piece belonging to X ”).

Remark.

- ▶ $E(S) = \mathbf{N}$;
- ▶ $I(S) = \mathcal{P}(\mathbf{N}) \setminus \{\mathbf{N}\}$;
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$\mathcal{S}$ is the combinatorial structures of all proper subsets of $\mathbf{N}$; setting $w(n) = n$, the mex statistic of the combinatorial structure $(\mathcal{S}, w)$ is the classical mex function from combinatorial game theory.

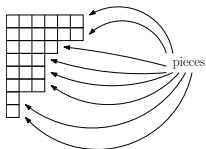
Computing the mex statistic

Theorem

Let (S, w) be a weighted combinatorial structure, in which a notion of size for its objects is also defined. Given $T \subseteq [m] = \{1, 2, \dots, m\}$, denote with $G_T(x)$ the generating function for the objects of S which do not contain any piece having weight i , for all $i \in T$ (with respect to the size). Then the generating function $\Gamma_m(x)$ for the objects of S having mex m (with respect to the size) is given by

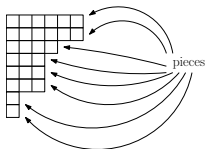
$$\Gamma_m(x) = \sum_{k=1}^m (-1)^{k+1} \sum_{\substack{T \subseteq [m] \\ |T|=k \\ m \in T}} G_T(x). \quad (1)$$

Integer partitions



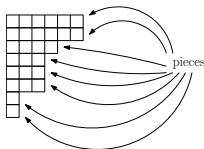
- ▶ $E(\mathcal{IP}) = \mathbf{N}$;
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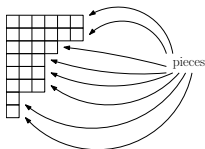
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$\mathcal{IP}$ is the combinatorial structure of *integer partitions*. With weight $w(n) = n$, $\text{mex}(\lambda)$ is defined as the smallest positive integer that is not a part of λ .

Integer partitions

A (more general) notion has been recently introduced [[Andrews and Newman, 2020](#)], but mostly studied from a number-theoretic perspective [[Barman and Singh, 2024](#)], [[da Silva and Sellers, 2020](#)], [[Hopkins, Sellers, and Stanton, 2022](#)], [[Kaur, Bhoria, Eyyunni, and Maji, 2022](#)].

Combinatorial aspects of our mex statistic were already studied (under the name “smallest gap in partitions”, [[Grabner and Knopfmacher, 2006](#)]), and the generating function for integer partitions having mex m (with respect to the size, i.e, the sum of the parts) is easily found:

$$\Gamma_m^{\mathcal{IP}}(x) = \left(x^{\binom{m}{2}} - x^{\binom{m+1}{2}} \right) P(x),$$

where $P(x) = \prod_{i \geq 1} \frac{1}{1-x^i}$ is the well known generating function of integer partitions (sequence A264401 in [[OEIS](#)]).

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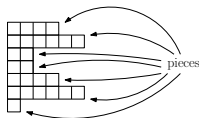
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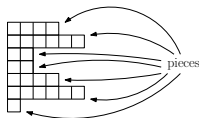
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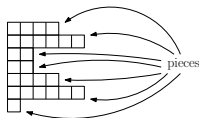
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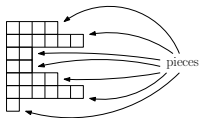
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Integer compositions

$G_T^{IC}(x)$: generating function of compositions having no parts equal to i , for all $i \in T \subseteq [m]$ (see also [Beck and Robbins, 2015]).

Nonempty compositions counted by $G_T^{IC}(x)$:

$$\underbrace{(\lambda_1)}_{\notin T}, \underbrace{(\lambda_2, \dots, \lambda_k)}_{\text{counted by } G_T^{IC}(x)}$$

$$G_T^{IC}(x) = 1 + H_T(x)G_T^{IC}(x),$$

where $H_T(x) = \frac{x}{1-x} - \sum_{i \in T} x^i$.

$$G_T^{IC}(x) = \frac{1}{1 - H_T(x)} = \frac{1 - x}{1 - 2x + \sum_{i \in T} (x^i(1 - x))}.$$

Proposition

The generating function $\Gamma_m^{IC}(x)$ of compositions having mex m is rational, for every $m \geq 1$.

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- ▶ $\Gamma_2^{IC}(x) = \frac{x(1-x)^2}{(1-2x+x^2-x^3)(1-x-x^3)}.$
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Integer compositions

$n \backslash m$	0	1	2	3	4	5	6	7
0	1							
1	0	0	1					
2	0	1	1					
3	0	1	1	2				
4	0	2	3	3				
5	0	3	6	7				
6	0	5	10	11	6			
7	0	8	18	26	12			
8	0	13	33	50	32			
9	0	21	59	104	72			
10	0	34	105	197	152	24		
11	0	55	187	387	335	60		
12	0	89	332	738	709	180		
13	0	144	588	1425	1489	450		
14	0	233	1040	2711	3092	1116		
15	0	377	1837	5170	6336	2544	120	
16	0	610	3241	9791	12894	5872	360	
17	0	987	5713	18543	26061	13032	1200	
18	0	1597	10063	34994	52380	28738	3300	
19	0	2584	17714	65990	104804	62208	8844	
20	0	4181	31166	124186	208843	133712	22200	
21	0	6765	54810	233505	414718	283938	54120	720

Table: $\gamma_{n,m}^{IC}$

Integer compositions

A general formula for $\gamma_{n,m}^{\mathcal{IC}}$:

- ▶ $n < \binom{m}{2}$: $\gamma_{n,m}^{\mathcal{IC}} = 0$;
- ▶ $n = \binom{m}{2} + k$, $k \geq 0$:

$$\gamma_{n,m}^{\mathcal{IC}} = \sum_{\substack{\alpha_1, \dots, \alpha_k \in \mathbb{N}, \alpha_m=0 \\ \alpha_1 + 2\alpha_2 + \dots + k\alpha_k = k}} \frac{(m-1 + \alpha_1 + \dots + \alpha_k)!}{\prod_{i=1}^{m-1} (\alpha_i + 1)! \prod_{i=m+1}^k \alpha_i!}.$$

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Inversion sequences

- ▶ $E(\mathcal{IS}) = \mathbf{N}$;
- ▶ $I(\mathcal{IS}) = \bigcup_{k \in \mathbf{N}} \mathbf{N}^k$;
- ▶ Properties: for $\mathbf{x} = (x_1, x_2, \dots, x_k) \in I(\mathcal{IS})$, $x_i \leq i - 1$, for all $i \leq n$.

Proposition

For $k \leq m \leq n$,

$$[x^n] \sum_{\substack{T \subseteq [m] \\ |T|=k \\ m \in T}} G_T^{IS}(\mathbf{x}) = (m - k + 1) S_{m, m-k+1}(n - k)!,$$

where the $S_{i,j}$'s are the Stirling numbers of the second kind.

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Proposition

For $k \leq m \leq n$,

$$[x^n] \sum_{\substack{T \subseteq [m] \\ |T|=k \\ m \in T}} G_T^{\mathcal{IS}}(\mathbf{x}) = (m - k + 1) S_{m, m-k+1}(n - k)!,$$

where the $S_{i,j}$'s are the Stirling numbers of the second kind.

Inversion sequences

- ▶ $E(\mathcal{IS}) = \mathbf{N}$;
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$\mathcal{IS}$ is the combinatorial structure of *inversion sequences*. With weight $w(n) = n$, $\text{mex}(\mathbf{x})$ is defined as the smallest positive integer that does not appear in $\mathbf{x}$.

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Fix $T = \{i_1, i_2, \dots, i_k = m\}$. How many choices for x_i ?

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x_2	2
$\dots$	$\dots$
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x_{i_1+2}	$i_1 + 1$
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x_{i_2}	$i_2 - 1$
x_{i_2+1}	$i_2 - 1$
x_{i_2+2}	i_2
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x_{i_k}	$i_k - (k - 1)$
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► number of inversion sequences without elements of T :

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$(m - k + 1)g(m, m - k + 1)(n - k)!$, where $g(n, k)$ is the sum of all possible products of $n - k$ (not necessarily distinct) integers in $[k]$;

$$\sum_{n \geq k} g(n, k) x^n = x^k \cdot \prod_{j=1}^k \frac{1}{1 - jx} = \sum_{n \geq k} S_{n,k} x^n$$

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Inversion sequences

Proposition

The coefficient of x^n in $\Gamma_m^{\mathcal{IS}}(x)$ is 0 when $n < m$, 1 when $n = m$, and

$$\sum_{k=1}^m (-1)^{k+1} (m - k + 1) S_{m, m-k+1} (n - k)!$$

when $n > m$.

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$m = 1$	$\gamma_{n,1}^{\mathcal{IS}} = (n - 1)!$
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$m = 3$	$\gamma_{n,3}^{\mathcal{IS}} = 3(n - 1)! - 6(n - 2)! + (n - 3)!$
$m = 4$	$\gamma_{n,4}^{\mathcal{IS}} = 4(n - 1)! - 18(n - 2)! + 14(n - 3)! - (n - 4)!$
$m = 5$	$\gamma_{n,5}^{\mathcal{IS}} = 5(n - 1)! - 40(n - 2)! + 75(n - 3)! - 30(n - 4)! + (n - 5)!$

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Apparently A056151 in [\[OEIS\]](#)?

Inversion sequences

Search: **a056151**

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page 1

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[A056151](#) **[Distribution of maximum inversion table entry.](#)**

+40

1, 1, 1, 1, 3, 2, 1, 7, 10, 6, 1, 15, 38, 42, 24, 1, 31, 130, 222, 216, 120, 1, 63, 422, 1050, 1464, 1320, 720, 1, 127, 1330, 4686, 8856, 10920, 9360, 5040, 1, 255, 4118, 20202, 50424, 80520, 91440, 75600, 40320, 1, 511, 12610, 85182, 276696, 558120, 795600, 851760, 685440, 362880

(list table graph refs links history raw internal format)

OFFSET

1,5

COMMENTS

$T(n,k)$ is the number of permutations p of $[n]$ such that $\max(p(i) - i) = k$. Example: $T(3,0) = 1$ because for $p = 123$ we have $\max(p(i) - i) = 0$; $T(3,1) = 3$ because for $p = 132, 213, 231$ we have $\max(p(i) - i) = 1$; $T(3,2) = 2$ because for $p = 312, 321$ we have $\max(p(i) - i) = 2$. - [Emeric Deutsch](#), Nov 12 2004

$T(n,k)$ is the number of permutations of $[n]$ for which the first subcedance occurs at position $n+2-k$. A subcedance of p_i occurs at position i if $i > p(i)$. For example, with $n = 3$ and $k = 2$, $T(n,k) = 3$ counts 132, 231, 321 in each of which the first subcedance occurs at position $n+2-k = 3$. - [David Callan](#), Dec 14 2021

REFERENCES

R. Sedgewick and Ph. Flajolet, "An Introduction to the Analysis of Algorithms", Addison-Wesley, 1996, ISBN 0-201-40009-X, table 6.10 (page 356)

LINES

Alois P. Heinz, [Rows n = 1..141, flattened](#)

Mathilde Bouvel, Lapo Cioni, and Luca Ferrari, [Preimages under the bubblesort operator](#), arXiv:2204.12936 [math.CO], 2022. See Table 1 p. 12.

Emeric Deutsch, Ira M. Gessel and David Callan, [Problem 18634: Permutation Parameters with the Same Distribution](#), Amer. Math. Monthly, 107 (2000), 567-568.

FORMULA

Lora R. Du and Guo-Niu Han, [The inversion number statistic for inversion sequences](#), arXiv:2604.17387 [math.CO], 2026. See p. 8.

Table[$\{(-1 + k) \cdot (1 - k + n) \cdot (-1 + k)!\} + k! \cdot (-k + n) \cdot k!$, $\{n, 1, 0\}$, $\{k, 1, n\}$]

$T(n, k) = k! \cdot (k!) \cdot (n - k) - (k - 1)! \cdot k! \cdot (n - k + 1)$ if $0 < k < n$; $T(n, 0) = 1$. - [Emeric Deutsch](#), Nov 12 2004

From [Peter Luschny](#), Dec 14 2021: (Start)

We assume T with offset 0.

$T(n, k) = k! \cdot 2^{n - (n - k)!} + [y^k] [x^n \cdot [n - k]] \cdot (\exp(\exp(x) \cdot y + x) \cdot (\exp(x) \cdot y - y + 1))$.

$T(n, k) = k! \cdot A343237(n, k)$. (End)

Inversion sequences

Permutations whose inversion sequences avoid and/or contain certain integers can be characterized in terms of mesh patterns.

- ▶ Inversion sequences having mex 1 correspond to permutations avoiding the mesh pattern



[Hilmarrsson, Jónsdóttir, Sigurðardóttir, Viðarsdóttir, Ulfarsson, 2015]

- ▶ More generally, inversion sequences having mex m correspond to permutations π having the following properties:
 - ▶ for all $1 \leq i \leq m - 1$, there exists a permutation τ_i of size i such that π contains the mesh pattern



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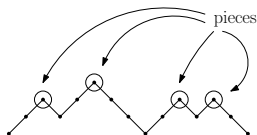
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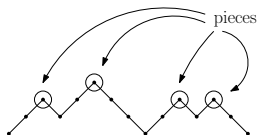


Dyck paths



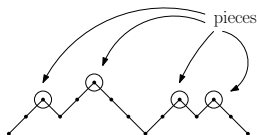
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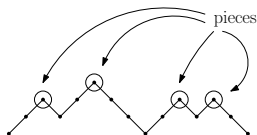
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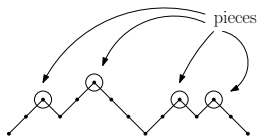
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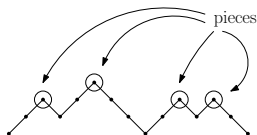
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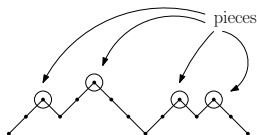
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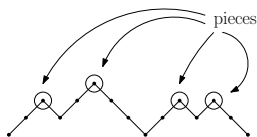
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Dyck paths



- ▶ $E(\mathcal{DP}) = \mathbf{N}^2$;
- ▶ $I(\mathcal{DP}) = \bigcup_{k \in \mathbf{N}} (\mathbf{N}^2)^k$;
- ▶ Properties: figure out those!
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$\mathcal{DP}$ is the combinatorial structure of *Dyck paths*. With weight $w(x, y) = y$, $\text{mex}(P)$ is defined as the smallest positive integer m such that P doesn't have peaks of height m .

Dyck paths

There is a continued fraction expression for the generating function of Dyck paths without peaks at prescribed heights [Eu, Lin, and Yeh, 2003]. Using our notations, for $T \subseteq \mathbf{N}$ with $\max T = m$,

$$G_T^{\mathcal{DP}}(x) = \frac{1}{1 + x\delta_T(1) - \frac{x}{1 + x\delta_T(2) - \frac{x}{\ddots \frac{x}{1 + x\delta_T(m-1) - \frac{x}{1 + x - xC(x)}}}}}$$

► $C(x)$ is the Catalan generating function;

► $\delta_T(i) = \begin{cases} 1, & i \in T \\ 0, & i \notin T \end{cases}$

Corollary

The generating function $\Gamma_m^{\mathcal{DP}}(x)$ is rational in x and $C(x)$, for all $m \geq 1$.

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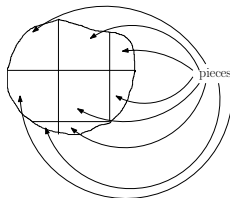
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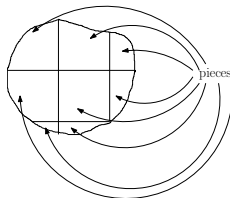
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Set partitions



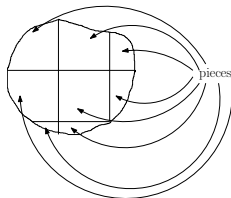
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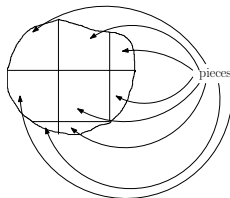
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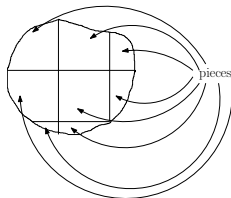
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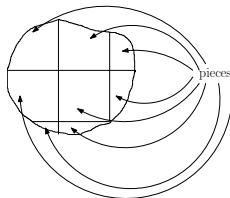
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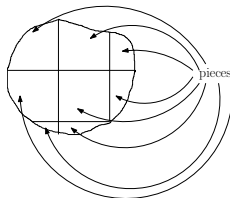
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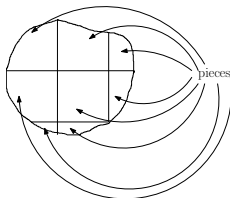
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$\mathcal{SP}$ is the combinatorial structure of *set partitions*. With weight $w(B) = |B|$, $\text{mex}(\pi)$ is defined as the smallest positive integer m such that π has no blocks of cardinality m .

Set partitions

Proposition

The exponential generating function $G_T^{SP}(x)$ is

$$G_T^{SP}(x) = e^{e^x - 1 - \sum_{n \in T} \frac{x^n}{n!}}.$$

Corollary

The exponential generating function $\Gamma_m^{SP}(x)$ of set partitions having mex m can be explicitly expressed as

$$\Gamma_m^{SP}(x) = e^{e^x - 1 - \frac{x^m}{m!}} \prod_{k=1}^{m-1} (1 - e^{-\frac{x^k}{k!}}).$$

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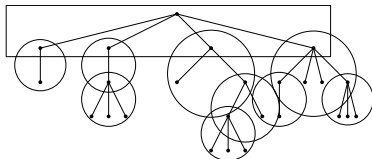
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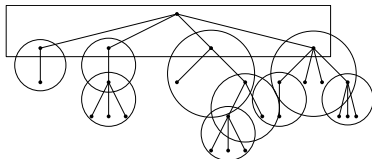
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Planar trees



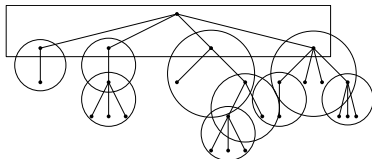
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Planar trees



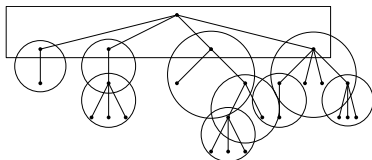
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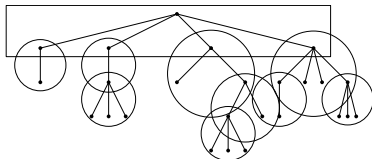
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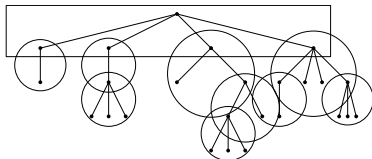
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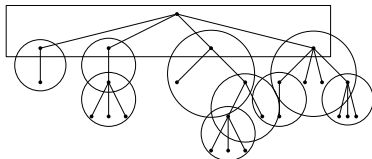
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$\mathcal{PT}$ is the combinatorial structure of *planar trees*. With weight $w(A) = |A| - 1$, $\text{mex}(\tau)$ is defined as the smallest positive integer m such that τ has no nodes having m children.

Planar trees

Proposition

Given $T \subseteq \mathbf{N}$, we get

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The generating function $\Gamma_m^{\mathcal{PT}}(x)$ of planar trees having mex m is algebraic.

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The generating function of planar trees having mex 1 is

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Using Lagrange inversion formula we also get:

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The n -th coefficient of the generating function $\Gamma_2^{\mathcal{PT}}(x)$ is

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thus obtaining the sequence 1,1,1,1,5,16,42,120,359,1078,3278, 10077, 31209, 97449, 306568,... which is not recorded in the [\[OEIS\]](#).

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What next?

- ▶ Interpretation of the mex statistic for inversion sequences in terms of the permutation statistics suggested in the [\[OEIS\]](#).
- ▶ mex in other combinatorial structures? Different mex on the same combinatorial structures?
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