Excedances and Deficiencies for Integer Partitions

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Excedances in permutations

An excedance of a permutation $\pi \in S_n$ is an index i for which $\pi(i) > i$. For example, in S_4 (using two-line notation),

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \\ + & & + & \end{bmatrix}$$

has two excedances, at $i = 1$ and $i = 3$.

Program: Permutation statistics applied to integer partitions.

Excedances in partitions

Write $P(n)$ for the partitions of n and $p(n) = |P(n)|$.
For $\lambda \in P(n)$, we list the parts λ_i in nonincreasing order.
A partition λ has an excedance for each i with $\lambda_i > i$. E.g.,

$$P(6) = \{\underset{+}{6}, \underset{+}{51}, \underset{+}{42}, \underset{+}{411}, \underset{++}{33}, \underset{+}{321}, \underset{+}{3111}, \underset{+}{222}, \underset{+}{2211}, \underset{+}{21111}, 111111\}$$

has

- 1 partition with no excedances (1^6),
- 9 partitions with one excedance, and
- 1 partitions with two excedances (33).

Triangle of partition excedances

n	0	1	2	3
0	1			
1	1			
2	1	1		
3	1	2		
4	1	4		
5	1	6		
6	1	9	1	
7	1	12	2	
8	1	16	5	
9	1	20	9	
10	1	25	16	
11	1	30	25	
12	1	36	39	1
13	1	42	56	2

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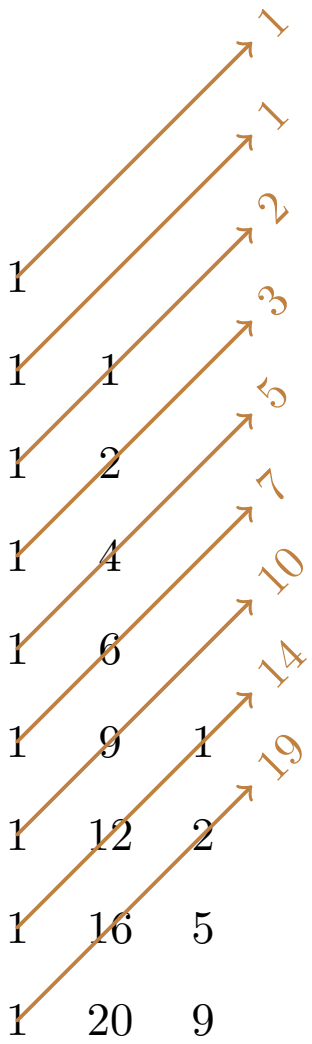
Number triangle questions:

- Sums: diagonals, rows, antidiagonals?
- Determine entries by higher rows, à la Pascal's lemma for binomial coefficients?

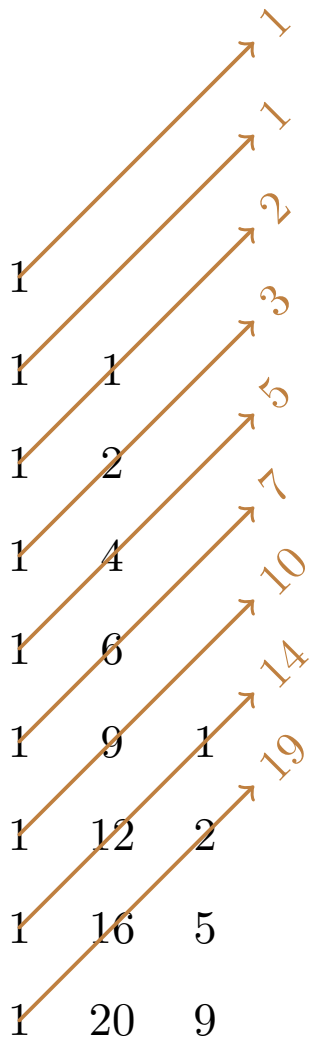
By construction, row sums are $p(n)$. And column k begins in row $k(k+1)$.

(OEIS A353318)

Excedance triangle diagonal sums



Excedance triangle diagonal sums



The sequence of diagonal sums continues

$$26, 35, 47, 62, 82, 107, 139,$$

which matches the partitions of n with positive crank.

Crank statistic was named with an intended application by Dyson in 1944, found by Andrews and Garvan in 1988.

Fixed points

Blecher & Knopfmacher (*Ramanujan J.* 2022) considered fixed points in integer partitions, $\lambda_i = i$.

$$P(6) = \{6, 51, 4\underline{=}2, 411, 33, 3\underline{=}21, 3111, 2\underline{=}2\underline{=}2, 2\underline{=}211, 21111, \underline{=}111111\}.$$

Note that $P(6)$ has 5 partitions with fixed points (at most one per partition), 6 without ($\sim$ derangements). They conjectured this inequality holds for all $n \geq 3$.

Fixed points

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H.–Sellers (*Discrete Math.* 2024)

Conjecture true; difference is the number of partitions of n with crank 0. Within the partitions of n, the following counts are equal:

- the number with a fixed point,
- the number with positive crank,
- the number with even mex (minimal excludant).

Connecting excedances and fixed points

Let $f(n, d)$ count the partitions of n with fixed point d.

$e(n, k) = f(n + k + 1, k + 1)$

Given $\lambda \in P(n)$ with k excedances, they are necessarily the first k parts. Note that $\lambda_k \geq k + 1$ and $\lambda_{k+1} \leq k + 1$ (else at least $k + 1$ excedances). Adding a $k + 1$ in the $(k + 1)$st position gives a partition of $n + k + 1$ with a fixed point in position $k + 1$.

For example,

$$P(6) = \{\underset{+}{6}, \underset{+}{51}, \underset{+}{42}, \underset{+}{411}, \underset{++}{33}, \underset{+}{321}, \underset{+}{3111}, \underset{+}{222}, \underset{+}{2211}, \underset{+}{21111}, 111111\}$$

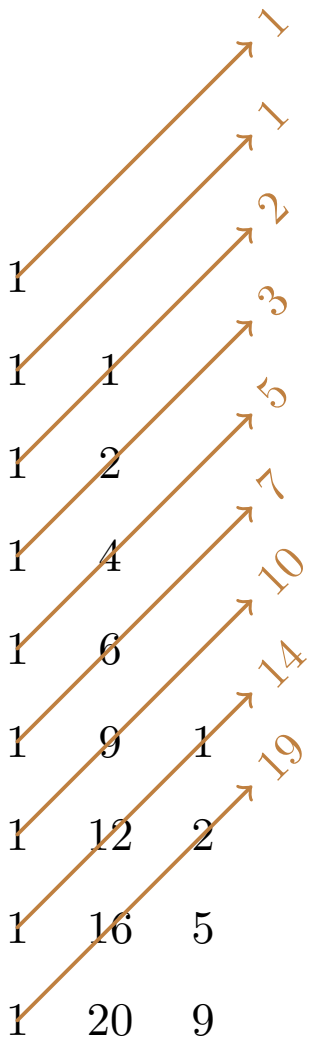
$$\mapsto \{\underset{+=}{62}, \underset{+=}{521}, \underset{+=}{422}, \underset{+=}{4211}, \underset{++=}{333}, \underset{+=}{3221}, \underset{+=}{32111},$$

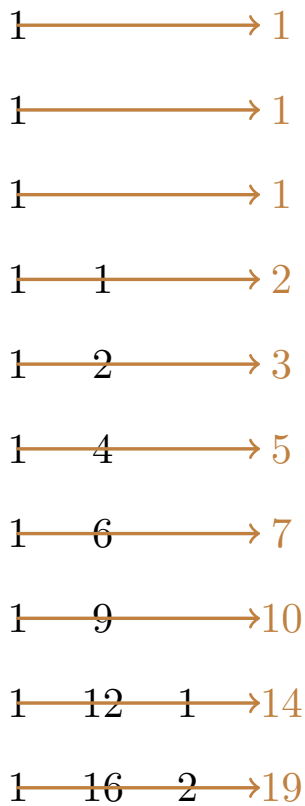
$$\underset{+=}{2222}, \underset{+=}{22211}, \underset{+=}{221111}, \underset{=}{1111111}\}.$$

Diagonals of the excedance triangle and fixed points

$E(8,0) \cup E(7,1) \cup E(6,2)$	$F(9,1) \cup F(9,2) \cup F(9,3)$
1^8	$\underline{1}11111111$
7	$7\underline{2}$
61	$6\underline{2}1$
52	$5\underline{2}2$
511	$5\underline{2}11$
421	$4\underline{2}21$
4111	$4\underline{2}111$
322	$3\underline{2}22$
3211	$3\underline{2}211$
31^4	$3\underline{2}1^4$
2221	$2\underline{2}221$
22111	$2\underline{2}2111$
21^5	$2\underline{2}1^5$
33	$33\underline{3}$

Connecting the excedance and fixed point triangles





Excedance triangle inter-column recurrence

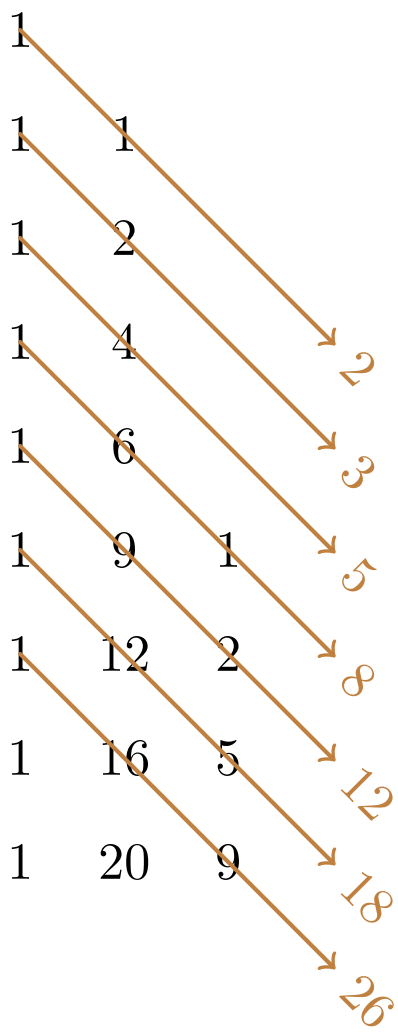
n	0	1	2	3
1	1			
2	1	1		
3	1	2		
4	1	4		
5	1	6		
6	1	9	-1	
7	1	12	2	
8	1	16	$+5$	
9	1	20	$+9$	
10	1	25	16	
11	1	30	$=25$	
12	1	36	39	1
13	1	42	56	2

From the analogous fixed point result (H., *Enumer. Combin. Appl.* 2024),

$$\begin{aligned} e(n,k) = {} & e(n-k,k) \\ & + e(n-k-1,k) \\ & - e(n-2k-1,k) \\ & + e(n-2k,k-1). \end{aligned}$$

E.g., $e(11,2) = 25$.

Excedance triangle antidiagonal sums

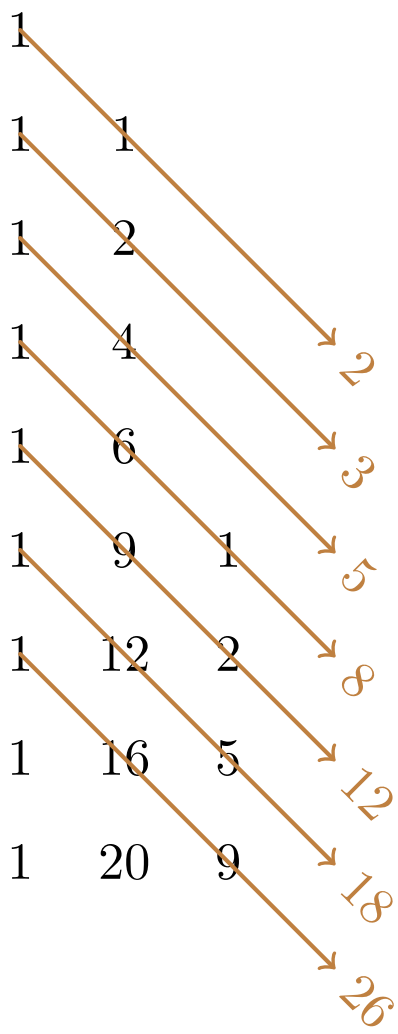


The sequence of diagonal sums continues

$$37, 52, 72, 98, 133, 178$$

matches $p(n-1) + p(n)$.

Excedance triangle antidiagonal sums



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matches $p(n-1) + p(n)$.

Interpret instead as a subset of overpartitions where a part 1 may be overlined.

Family of related triangles

The same connection between the triangles counting fixed points and excedances generates a family of triangles with the same columns having different offsets. For each,

- What do the row sums count?
- How are those objects partitioned into column entries?
- Connection to the fixed point triangle gives an inter-column recurrence.
- Diagonal and antidiagonal sums match row sums of "adjacent" triangles.

Some other objects counted by rows of related triangles:

- partitions whose parts do not include the Durfee square* size,
- overpartitions where only 1 can be overlined,
- unimodal compositions (can be described with overlining).

Deficiencies in permutations

An deficiency (or antiexcedance or nonexcedance) of a permutation $\pi \in S_n$ is an index i for which $\pi(i) < i$. For example, in S_4,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & \underline{1} \end{bmatrix}$$

has one deficiency, at $i = 4$.

Deficiencies in permutations

An deficiency (or antiexcedance or nonexcedance) of a permutation $\pi \in S_n$ is an index i for which $\pi(i) < i$. For example, in S_4,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & \underset{-}{1} \end{bmatrix}$$

has one deficiency, at $i = 4$.

Among all permutation of n, there are an equal number of excedances and deficiences as the counts are swapped in inverses.

$$\text{E.g., } \begin{bmatrix} 1 & 2 & 3 & 4 \\ \underset{+}{3} & 2 & \underset{+}{4} & \underset{-}{1} \end{bmatrix} \text{ has inverse } \begin{bmatrix} 1 & 2 & 3 & 4 \\ \underset{+}{4} & 2 & \underset{-}{1} & \underset{-}{3} \end{bmatrix}.$$

Deficiencies in partitions

That balancing out does not happen for partitions.

$$P(6) = \{\underset{+}{6}, \underset{+-}{51}, \underset{+}{42}, \underset{+--}{411}, \underset{++}{33}, \underset{+-}{321}, \underset{+---}{3111}, \underset{+}{222}, \underset{+-}{2211}, \underset{+----}{21111}, \underset{-----}{111111}\}.$$

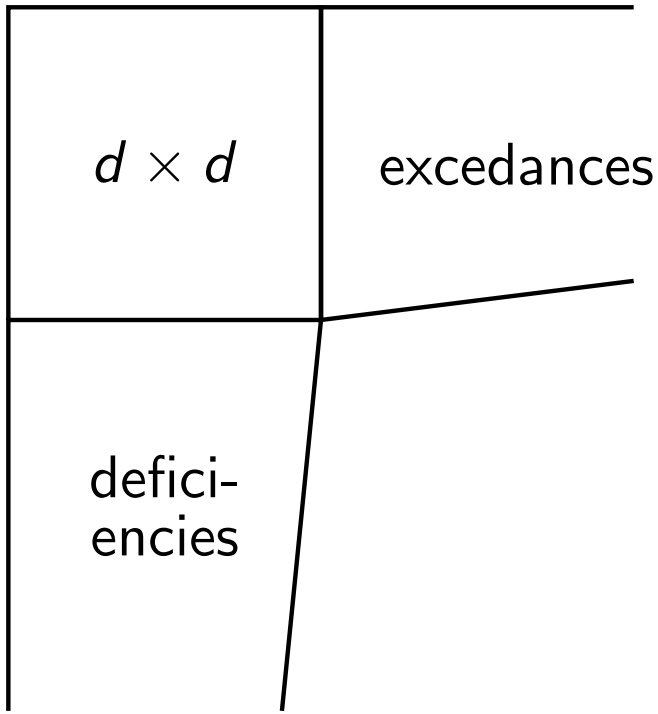
For example, $P(6)$ has 8 more deficiencies than excedances.

Deficiency counts in $P(6)$:

- 3 partitions with no deficiencies (6, 42, 33),
- 3 partitions with one deficiency (51, 321, 222),
- 2 partitions with two deficiencies (411, 2211), and
- 1 partition each with 3, 4, 5 deficiencies.

Durfee square, excedances, and deficiencies

The Durfee square of $\lambda \in P(n)$ is the greatest index d for which $\lambda_d \geq d$; gives a square of d^2 dots in the Ferrers diagram.



A partition with Durfee square d has $d-1$ or d excedances.

There is no structural limit on the number of deficiences (could have $n - d^2$ parts 1).

Triangle of partition deficiencies

n	0	1	2	3	4	5	6	7	8
1	1								
2	1	1							
3	1	1	1						
4	2	1	1	1					
5	2	2	1	1	1				
6	3	3	2	1	1	1			
7	3	4	3	2	1	1	1		
8	4	5	5	3	2	1	1	1	
9	5	6	6	5	3	2	1	1	1

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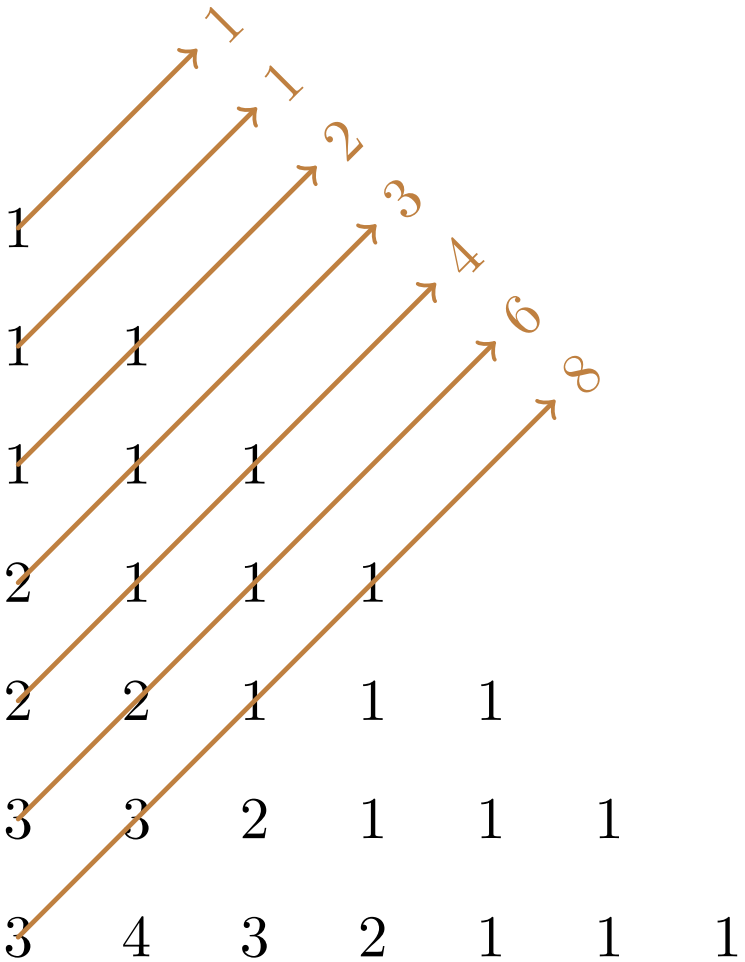
5	2	2	1	1	1				
6	3	3	2	1	1	1			
7	3	4	3	2	1	1	1		
8	4	5	5	3	2	1	1	1	
9	5	6	6	5	3	2	1	1	1

First column is the number of Rogers–Ramanujan partitions (parts congruent to 1 or 4 modulo 5), second column has a partition interpretation.

No finite inter-column recurrence, infinite anti-diagonal sums.

Deficiency triangle diagonal sums



The sequence of diagonal sums continues

$$11, 15, 20, 26, 35, 45, 58$$

matches the partitions of n with no part 2.

With Dhar and Sellers, fixed points of some restricted partitions. Lots more permutation statistics to explore.