

# Bijections between pattern-avoiding derangements and desarrangements

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# Derangements and Fixed Points

We denote the symmetric group of length  $n$  permutations as  $\mathfrak{S}_n$  and the set of length  $n$  permutations which avoid every pattern  $\sigma \in \Pi$  as  $\mathfrak{S}_n(\Pi)$ .

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Given a permutation  $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$  of length  $n$ , a **fixed point** of  $\pi$  is any position  $i \in [n]$  for which  $\pi_i = i$ .

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For instance, Robertson–Saracino–Zeilberger (2002) and Mansour–Robertson (2002) found the distribution of  $\text{fix}$  over all  $\mathfrak{S}_n(\Pi)$ , where  $\Pi \subseteq \mathfrak{S}_3$ .

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Désarménien (1984) introduced desarrangements to give a combinatorial proof of the recurrence

$$d_n = nd_{n-1} + (-1)^n.$$

# Desarrangements and Pixed Points

Given a permutation  $\pi = \pi_1\pi_2 \cdots \pi_n$  of length  $n$ , a **descent** of  $\pi$  is any position  $i \in [n - 1]$  for which  $\pi_i > \pi_{i+1}$ .

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## Example

Let  $\pi = 41652378$ . Then, the descents of  $\pi$  are 1, 3, and 4. The non-descents of  $\pi$  are 2, 5, 6, 7, and 8.

Notice that  $n$  is always a non-descent.

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A **desarrangement** is a permutation in which the first non-descent is even.

# Desarrangements and Pixed Points

Let  $\mathfrak{D}_n$  denote the set of desarrangements of length  $n$  and  $\mathfrak{D}_n(\Pi)$  denote the set of desarrangements of length  $n$  which avoid every pattern  $\sigma \in \Pi$ .

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As it turns out,  $|\mathfrak{D}_n| = |\mathcal{D}_n|$  for all  $n \geq 0$ . For which  $\Pi \subseteq \mathfrak{S}_3$  do we have  $|\mathfrak{D}_n(\Pi)| = |\mathcal{D}_n(\Pi)|$  for all  $n \geq 0$ ?

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## Theorem (Bsila–Cox–Hugo–Styron–Zhuang 2026)

*If  $\Pi \subseteq \mathfrak{S}_3$  and  $1 \leq |\Pi| \leq 3$ , then  $|\mathfrak{D}_n(\Pi)| = |\mathcal{D}_n(\Pi)|$  for all  $n \geq 0$  precisely when  $\Pi \in \{\{132\}, \{132, 312\}, \{132, 321\}, \{213, 231\}, \{123, 132, 312\}, \{123, 213, 231\}, \{123, 312, 321\}, \{132, 312, 321\}, \{213, 231, 312\}, \{213, 231, 321\}\}$ .*

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Foata–Han (2008) showed that every permutation  $\pi$  can be uniquely decomposed as  $\pi = \iota\delta$ , where  $\iota$  is an increasing permutation and  $\delta$  is a desarrangement. This is called the pixed factorization of  $\pi$ , and the letters of  $\iota$  are the pixed points of  $\pi$ .

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## Example

Let  $\pi = 1256473$ . Notice that  $\pi = 125 \quad 6473$ . So,  $\iota = 125$  and  $\pi$  has 3 pixed points.

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The permutation statistic

$$\text{pix}(\pi) = |\iota| \quad \text{where} \quad \pi = \iota\delta$$

is equidistributed with  $\text{fix}$  in the full symmetric group  $\mathfrak{S}_n$ .

# The Conjecture and the Theorem

For which  $\Pi \subseteq \mathfrak{S}_3$  does the equinumerosity between  $\mathcal{D}_n(\Pi)$  and  $\mathfrak{D}_n(\Pi)$  extend to an equidistribution between  $\text{fix}$  and  $\text{pix}$  over all of  $\mathfrak{S}_n(\Pi)$ ?

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Recently, Dong–Xu (2026) and Henke–S.–Zhuang (2026) independently proved a strengthened version of this conjecture.

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Let us begin with a  $\text{des}$ -preserving bijection between  $\mathcal{D}_n(213, 231)$  and  $\mathfrak{D}_n(213, 231)$ .

# 213-231-avoiding Permutations and ad-words

We exploit the fact that every permutation  $\pi \in \mathfrak{S}_n(213, 231)$  is uniquely determined by its ascents and descents.

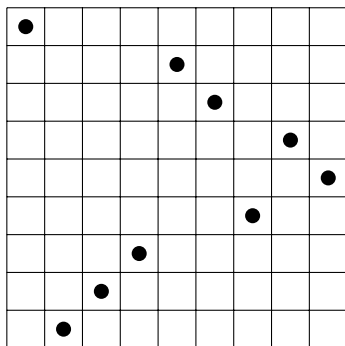


**Figure:** The permutation array of a typical 213-231-avoiding permutation.

# 213-231-avoiding Permutations and ad-words

We can convert any 213-231-avoiding permutation into its string of ascents and descents, called its **ad-word**.

## Example

Start with a 213-231-avoiding derangement  $\pi \in \mathcal{D}_n(213, 231)$ . We can take  $\pi = 11\ 10\ 9\ 1\ 2\ 8\ 3\ 7\ 6\ 4\ 5$ . Then,

$$\begin{array}{cccccccccccc} \pi & = & 11 & 10 & 9 & 1 & 2 & 8 & 3 & 7 & 6 & 4 & 5 \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{ad}(\pi) & = & D & D & D & A & A & D & A & D & D & A & \end{array}$$

The last letter of  $\pi$  does not get a respective entry in  $\text{ad}(\pi)$  since it can be recovered from the previous entries.

# The Initial Bijection $\mathcal{D}_n(213, 231) \rightarrow \mathfrak{D}_n(213, 231)$

There are two things we know about this ad-word, since it originated from a derangement.

- 1 The first letter is D.
- 2 Not counting the first letter, there exists one position where the number of Ds to the left is equal to the number of letters on the right.

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The second condition allows us to use the right half of the ad-word as a sort of “key” to encode the left half of the ad-word.

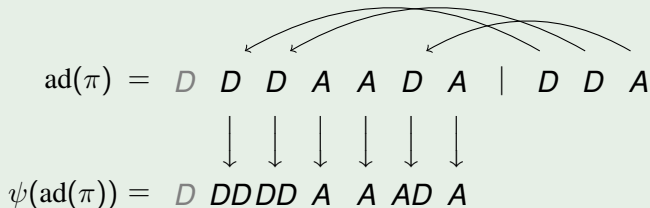
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# The Initial Bijection $\mathcal{D}_n(213, 231) \rightarrow \mathfrak{D}_n(213, 231)$

The first  $A$  will be preceeded by an even number of  $D$ s. After prepending a  $D$ , the corresponding 213-231-avoiding permutation will be a desarrangement.

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|--------------------------------|---|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|---|
| $\psi(\text{ad}(\pi))$         | = | $D$          | $D$          | $D$          | $D$          | $D$          | $A$          | $A$          | $A$          | $D$          | $A$          |   |
|                                |   | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |   |
| $\theta(\psi(\text{ad}(\pi)))$ | = | 11           | 10           | 9            | 8            | 7            | 1            | 2            | 3            | 6            | 4            | 5 |

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Notice that the bijection  $\psi$  preserves the number of  $D$ s in each ad-word. Thus, the bijection from  $\mathcal{D}_n(213, 231)$  to  $\mathfrak{D}_n(213, 231)$  is des-preserving.

# The Modified Bijection $\hat{\mathcal{D}}_n(213, 231) \rightarrow \hat{\mathfrak{D}}_n(213, 231)$

Let  $\hat{\mathcal{D}}_n(213, 231)$  denote the set of permutations in  $\mathfrak{S}_n(213, 231)$  with 1 descent fixed point.

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This bijection will actually be quite similar to our previous one.

# The Modified Bijection $\widehat{\mathcal{D}}_n(213, 231) \rightarrow \widehat{\mathfrak{S}}_n(213, 231)$

- 1 Convert a permutation  $\pi \in \widehat{\mathcal{D}}_n(213, 231)$  to its ad-word.

## Example

$$\begin{array}{rcccccccccccc} \pi & = & 11 & 10 & 1 & 2 & 3 & 9 & 4 & 8 & 7 & 5 & 6 \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{ad}(\pi) & = & D & D & A & A & A & D & A & D & D & A & \end{array}$$

# The Modified Bijection $\hat{\mathcal{D}}_n(213, 231) \rightarrow \hat{\mathcal{D}}_n(213, 231)$

- 2 Remove the initial  $D$ , and the  $D$  which corresponds to a fixed point.

## Example

$$\begin{array}{rcccccccccccc} \pi & = & 11 & 10 & 1 & 2 & 3 & 9 & 4 & 8 & 7 & 5 & 6 \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{ad}(\pi) & = & D & D & A & A & A & D & A & D & D & A & \end{array}$$

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- 3 Use the right side of the ad-word as a key to encode the left side.

## Example

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- 4 Prepend one of the  $D$ s to turn the resulting permutation into a desarrangement, and the other to add a descent fixed point.

## Example

$$\begin{array}{rcl}
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 & & \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \\
 \text{ad}(\pi) & = & D \ D \ A \ A \ A \ D \ A \ D \ D \ A \\
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 \hat{\psi}(\text{ad}(\pi)) & = & DDDD \ A \ A \ A \ AD \ A
 \end{array}$$

# The Modified Bijection $\hat{\mathcal{D}}_n(213, 231) \rightarrow \hat{\mathfrak{D}}_n(213, 231)$

- Convert the resulting ad-word back into a 213-231-avoiding permutation.

## Example

$$\begin{array}{rcl}
 \pi & = & 11 \ 10 \ 1 \ 2 \ 3 \ 9 \ 4 \ 8 \ 7 \ 5 \ 6 \\
 & & \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \\
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 & & \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \\
 \hat{\psi}(\text{ad}(\pi)) & = & D D D D \ A \ A \ A \ A D \ A \\
 & & \swarrow \searrow \swarrow \searrow \swarrow \searrow \swarrow \searrow \swarrow \searrow \\
 \theta(\hat{\psi}(\text{ad}(\pi))) & = & 11 \ 10 \ 9 \ 8 \ 1 \ 2 \ 3 \ 4 \ 7 \ 5 \ 6
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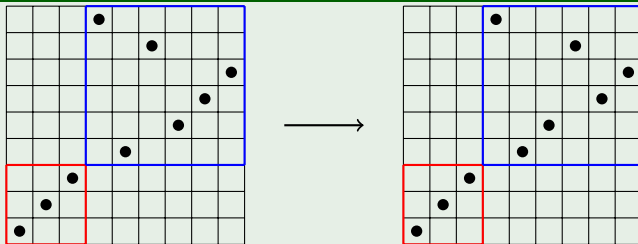
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We can use our two previous bijections depending on which case we are in.

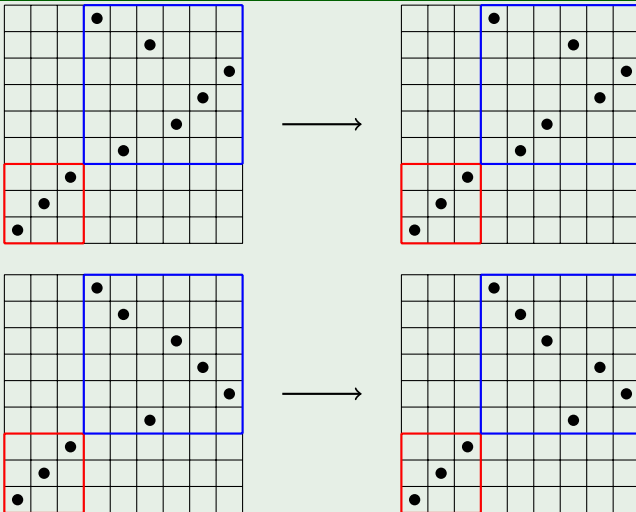
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We can use our complete bijection  $\Psi$  to achieve equidistributions of  $(\text{fix}, \text{des})$  and  $(\text{pix}, \text{des})$  for other sets of patterns.

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- The reverse-complement involution

$$\text{rc}(\pi) := (n - \pi_n + 1)(n - \pi_{n-1} + 1) \cdots (n - \pi_1 + 1)$$

preserves  $\text{fix}$ , preserves  $\text{des}$ , and sends a permutation in  $\mathfrak{S}_n(132, 312)$  to a permutation in  $\mathfrak{S}_n(213, 231)$ .

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$$\text{rc}(\pi) := (n - \pi_n + 1)(n - \pi_{n-1} + 1) \cdots (n - \pi_1 + 1)$$

preserves  $\text{fix}$ , preserves  $\text{des}$ , and sends a permutation in  $\mathfrak{S}_n(132, 312)$  to a permutation in  $\mathfrak{S}_n(213, 231)$ .

- A permutation  $\pi \in \mathfrak{S}_n(213, 231)$  avoids 321 if and only if  $\text{des}(\pi) \leq 1$ . Since  $\Psi$  preserves  $\text{des}$ , applying  $\Psi$  also preserves the property of avoiding 321. The same is true of the pattern 123.

# Using $\Psi$

We can use our complete bijection  $\Psi$  to achieve equidistributions of  $(\text{fix}, \text{des})$  and  $(\text{pix}, \text{des})$  for other sets of patterns.

- The reverse-complement involution

$$\text{rc}(\pi) := (n - \pi_n + 1)(n - \pi_{n-1} + 1) \cdots (n - \pi_1 + 1)$$

preserves  $\text{fix}$ , preserves  $\text{des}$ , and sends a permutation in  $\mathfrak{S}_n(132, 312)$  to a permutation in  $\mathfrak{S}_n(213, 231)$ .

- A permutation  $\pi \in \mathfrak{S}_n(213, 231)$  avoids 321 if and only if  $\text{des}(\pi) \leq 1$ . Since  $\Psi$  preserves  $\text{des}$ , applying  $\Psi$  also preserves the property of avoiding 321. The same is true of the pattern 123.
- The analogous facts also hold true for  $\mathfrak{S}_n(132, 312)$ .

Thank you!

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