From Consecutive Avoidance to Prescribed Indices

local constraints, global structures, and open problems

Mohamed Omar (York University)

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Let $\pi \in \mathfrak{S}_k$, let $\sigma \in \mathfrak{S}_n$. Record

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Consecutive-pattern avoidance: $I = \emptyset$. Past participants in the Permutation Patterns community helped build it.

Some conference voices	**What they helped develop**
Babson–Steingrímsson; Claesson; Kitaev	generalized and vincular patterns
Elizalde–Noy; Ehrenborg–Kitaev–Perry	enumeration and asymptotics
Dotsenko–Khoroshkin; Sagan; Billey and collaborators	algebra, bijections, and refinements

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$\pi = \mathbf{21}$, $\sigma =$

$$\underset{1}{2} < \underset{2}{4} > \underset{3}{3} > \underset{4}{1} < \underset{5}{5} < \underset{6}{6} < \underset{7}{7} < \underset{8}{8}$$

Our running question: *where* does the adjacent pattern **21** occur?

Permutation Patterns conference history; selected references listed at the end.

The new datum is simply the set of starts

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$$\pi = \mathbf{21},\ \sigma = \underset{1}{2} < \underset{2}{4} > \underset{3}{3} > \underset{4}{1} < \underset{5}{5} < \underset{6}{6} < \underset{7}{7} < \underset{8}{8}$$

Here $\mathsf{std}(43) = \mathsf{std}(31) = \mathbf{21}$, and no other adjacent pair reduces to $\mathbf{21}$. Therefore

$$I = \{2, 3\}.$$

QUESTION: How many $\sigma \in \mathfrak{S}_n$ have a (or several) fixed permutation(s) PRECISELY at I?

Where fixed-index questions have appeared in the literature

Data prescribed	Quantity studied	Representative development
no starting position realizes π	avoidance sequence	clusters, generating functions, asymptotics
finite descent set I	polynomial $d(I;n)$	Omar et al.; then posets, tableaux, q-weights
finite peak set S	$2^{n-\lvert S\rvert-1}p_S(n)$	Omar et al.; then flips, lattices, q-weights
peak vertices of a graph	number of labelings	Omar et al.; then graph pinnacles and posets
an infinite binary profile	exponential growth rate	Omar–Troyka
an X-descent set	word and path counts	Omar

Five fixed profiles lead to five unexpected toolkits

Each panel pairs a fixed-index question with the toolkit it brings into the story.

descents	**peaks**	**graphs**	**words**	X**-descents**
Which coefficients count?	Why are differences positive?	What replaces the path?	What if the data never ends?	What if the rule changes?
positive bases, posets	**differences, flips**	**deletion, independent sets**	**growth-rate spectra**	**Hamiltonian paths**

The thesis: local positional data releases combinatorics across the field.

ACT I: DESCENTS

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Example: For $I = \{1, 2\}$, the word "exactly" requires

$$\sigma(1) > \sigma(2) > \sigma(3) < \sigma(4) < \cdots < \sigma(n).$$

The valley must contain 1. Choose the other two entries of the decreasing prefix from $\{2, \ldots, n\}$; their order and the increasing suffix are forced.

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$$d(\{1,2\};n) = \binom{n-1}{2}.$$

finite label choice + **forced relative order** = **polynomial count** .

MacMahon; example as presented in Díaz-Lopez et al. (2019).

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Example $m = 3$: the suffix beginning at position 4 is increasing.

$$\sigma = \underbrace{\mathbf{243}}_{\text{first } m \text{ positions}} \Big| \underbrace{\mathbf{15678}}_{\text{increasing suffix}} .$$

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As n grows, the comparison pattern stays fixed; only the available labels change.

A bounded comparison pattern plus growing labels naturally produces a polynomial.

The monomial basis hides the positivity

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The first natural question is about its ordinary coefficients:

$$d(I;n) = c_0 + c_1 n + \cdots + c_m n^m.$$

They need not be nonnegative. Our first example already gives

$$d(\{1,2\};n) = \binom{n-1}{2} = \tfrac{1}{2}n^2 - \tfrac{3}{2}n + 1.$$

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Díaz-Lopez, Harris, Insko, Omar, and Sagan found the positive basis:

$$d(I;n) = \sum_{k=0}^{m} a_k(I)\binom{n-m}{k}, \qquad a_k(I) \geq 0.$$

The prescribed positions determine where the polynomial should be centered.

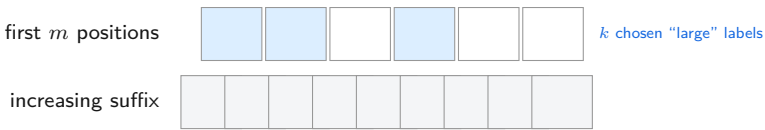
MacMahon; Díaz-Lopez–Harris–Insko–Omar–Sagan, *Descent polynomials* (2019).

The positive binomial basis has a direct counting meaning

For $\sigma \in D(I;n)$ and again $m = \max(I)$, let

$$k = \#\Big(\{\sigma(1),\dots,\sigma(m)\} \cap \{m+1,\dots,n\}\Big).$$

Thus k is the number of "large" labels used in the bounded prefix.



Choose those labels in $\binom{n-m}{k}$ ways; after standardizing, $a_k(I)$ possible relative-order shapes remain. Hence

$$d(I;n) = \sum_{k=0}^{m} a_k(I)\binom{n-m}{k}.$$

For example, $d(\{1,2\};n) = \binom{n-1}{2} = \binom{n-2}{1} + \binom{n-2}{2}$, so $(a_0,a_1,a_2) = (0,1,1)$.

Every summand counts a disjoint family, so positivity is visible.

Díaz-Lopez–Harris–Insko–Omar–Sagan (2019), Theorem 3.3.

Nonnegative coefficients are a beginning, not an endpoint

Fix $I \neq \varnothing$ and let $m = \max(I)$. Then

$$d(I;n) = \sum_{k=0}^{m} a_k(I)\binom{n-m}{k}, \qquad a_k(I) \geq 0$$

What do combinatorialists usually ask once positivity appears?

Nonnegative coefficients are a beginning, not an endpoint

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What do combinatorialists usually ask once positivity appears?

- **INTERPRET** **What do the coefficients count?**
  Find natural objects $\mathcal{A}_k(I)$ and a direct formula or bijection $a_k(I) = |\mathcal{A}_k(I)|$.
- **FIND THE SHAPE** **Is the sequence unimodal?**
  Do the coefficients increase to one peak and then decrease?
- **STRENGTHEN THE SHAPE** **Is the sequence log-concave?**
  That is, does $a_k(I)^2 \geq a_{k-1}(I)a_{k+1}(I)$ hold for every relevant k?

Díaz–Lopez–Harris–Insko–Omar–Sagan, *Descent polynomials*, Theorem 3.3.

Descent polynomials: coefficients and roots

Fix $I \subseteq \mathbb{Z}_{>0}$, let $m = \max(I \cup \{0\})$, and let

$$d(I;n) = \#\{\pi \in \mathfrak{S}_n : \mathrm{Des}(\pi) = I\}.$$

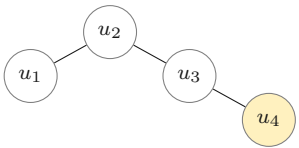
Díaz–López, Harris, Insko, Omar, and Sagan studied the algebraic shape of this polynomial.

Question	**Finding for $d(I;n)$**	**Message**
What do the coefficients count?	$a_k(I)$ counts permutations in $D(I;2m)$ whose first m entries contain exactly the interval $[m+1, m+k]$ from the upper half of $[2m]$.	Coefficient positivity has a direct combinatorial explanation.
Do the coefficients have shape?	For interval descent sets $I = \{\ell, \ell+1, \ldots, m\}$, the sequence $(a_k(I))_{k=0}^{m}$ is log-concave.	This connects descent polynomials to the broader log-concavity story.
Which roots are forced?	Every prescribed descent index is a root: $i \in I \implies d(I;i) = 0$.	The index set I is visible algebraically.
Where can the other roots go?	$z_0 \in \mathbb{R},\ d(I;z_0) = 0 \implies z_0 \leq 2m-1$.	The remaining roots are constrained, not chaotic.

A. Díaz–López, P. E. Harris, E. Insko, M. Omar, and B. E. Sagan, *Descent polynomials*, Discrete Math. 342 (2019), 1674–1686.

Bencs: Log-concavity via Posets

For $I = \{2, 3\}$, in $\mathfrak{S}_4$ the required comparisons are $\sigma_1 < \sigma_2 > \sigma_3 > \sigma_4$.



$$P_I : \quad u_1 < u_2 > u_3 > u_4$$

Vertices are positions; height respects $>$.

A linear extension is a legal labeling.
Assign $1, 2, 3, 4$ to the vertices so every displayed inequality is respected. Reading the labels from left to right gives a permutation with descent set exactly I:

$$\begin{matrix} 2 & 4 & 3 & 1 \\ u_1 & u_2 & u_3 & u_4 \end{matrix} \iff 2431 \in D(\{2,3\}; 4).$$

There are three legal labelings. Grouped by the label on u_4:

$$\begin{matrix} u_4 = 1 & : & 2431,\ 3421, \\ u_4 = 2 & : & 1432 \end{matrix} \implies (2, 1, 0, 0).$$

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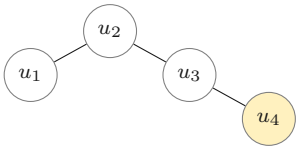

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Why this helps. For general I, let h_j count the linear extensions in which exactly j labels occur below the endpoint u_{m+1}. Stanley's height theorem says that

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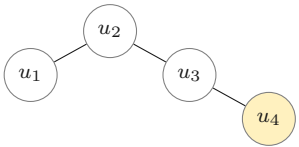

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Why this helps. For general I, let h_j count the linear extensions in which exactly j labels occur below the endpoint u_{m+1}. Stanley's height theorem says that

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is log-concave. Bencs proves that $(a_k(I))_{k=0}^m$ is obtained from these endpoint counts by reversing them and applying a binomial transformation that preserves log-concavity. Hence $(a_k(I))_{k=0}^m$ is log-concave.

Bencs, *Some coefficient sequences related to the descent polynomial*, Theorem 2.3 and Proposition 4.1–Corollary 4.4.

Later papers keep the profile and change the model

Once the exact-descent fiber had positive and order-theoretic models, later authors changed one ingredient at a time and asked what survived.

What is changed?	What is retained?	Authors
Add the weight $q^{\mathrm{inv}(\sigma)}$.	Keep the prescribed descent set I.	Gaetz–Gao
Allow repeated labels.	Keep a fixed comparison profile.	Raychev
Replace permutations by ribbon tableaux.	Keep the same descent data.	Jiradilok–McConville
Count only selected inversions.	Keep the fiber while changing its grading.	Lee–Lesnevich–Precup

The original fiber became a portable object: grade it, repeat labels, or change the carrier.

Each extension keeps the prescribed comparison profile and asks which structural properties survive.

Two familiar phenomena lead to open questions

q-refinement Weight the descent fiber by inversions:

$$D_I(n;q) = \sum_{\mathrm{Des}(\sigma)=I} q^{\mathrm{inv}(\sigma)} = \sum_{k=0}^{m} a_k(I;q) \begin{bmatrix} n-m \\ k \end{bmatrix}_q .$$

Here $(a_k(I;q))_k$ is the sequence $(a_0(I;q), \ldots, a_m(I;q))$.

OPEN **Gaetz–Gao, Conjecture 7:** Is it strongly q-log-concave?

That is, every $a_i a_j - a_{i-1} a_{j+1}$ is coefficientwise nonnegative.

negative evaluation Negative evaluations often count hidden objects; for example,

$$(-1)^{|V(G)|} \chi_G(-1) = \#\{\text{acyclic orientations of } G\}.$$

Bencs proved $c_k(I) \geq 0$ in

$$d(I;n) = \sum_{k=0}^{m} a_k(I) \binom{n-m}{k}$$

OPEN **Bencs, Question 6.1:** What do $c_k(I)$ and $(-1)^m d(I;-n)$ count?

All this rich theory came from asking about descent polynomials.

Gaetz–Gao, Conj. 7; Bencs, Question 6.1; Stanley's chromatic reciprocity.

ACT II: PEAKS

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For $\sigma = \mathbf{24315678}$, the descent word is $\mathbf{0110000}$. A peak is a $0 \to 1$ transition in this word, so

$$\mathrm{Des}(\sigma) = \{2, 3\}, \qquad \mathrm{Peak}(\sigma) = \{2\}.$$

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A peak remembers only the *first* descent in a consecutive block of descents.

An admissible peak set excludes 1 and contains no consecutive positions.

What does the running peak set $S = \{2\}$ require?

$S = \{2\}$ means:

- rise into position 2, then fall;
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$$\#\{\sigma \in \mathfrak{S}_n : \mathrm{Peak}(\sigma) = S\} = p_S(n)\, 2^{n-|S|-1}, \qquad \deg p_S = m - 1.$$

For $S = \{2\}$ this becomes

$$p_{\{2\}}(n) = n-2, \qquad \#\{\sigma \in \mathfrak{S}_n : \mathrm{Peak}(\sigma) = \{2\}\} = (n-2)2^{n-2} \qquad (\#\text{ in } \mathfrak{S}_n = \#\text{ in } \mathfrak{S}_{n-1} + (n-1)2^{n-3}).$$

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the only peak

$$\begin{array}{ccccccccccccccc} 2 & < & 4 & > & 3 & > & 1 & < & 5 & < & 6 & < & 7 & < & 8 \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 \end{array}$$

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Which basis makes the coefficients of p_S visibly positive?

Billey–Burdzy–Sagan, *Permutations with given peak set* (2013).

Díaz-Lopez–Harris–Insko–Omar: the theorem behind the example

For every admissible S, with $m = \max S$, they prove

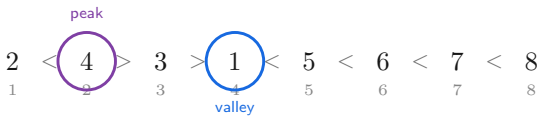
$$p_S(n) = \sum_{j=0}^{m-1} (\Delta^j p_S)(m) \binom{n-m}{j}, \qquad (\Delta^j p_S)(m) \geq 0.$$

Proof idea: similar to descent idea, but finicky.

Proof mechanism in Díaz-Lopez–Harris–Insko–Omar (2017).

Oğuz: the beginnings and ends of descent runs become peak pieces

For a descent set D, let $\mathrm{Spike}(D)$ record the beginnings and ends of its descent runs. In the running example, $D = \{2, 3\}$ has a peak at 2 and a valley at 4.



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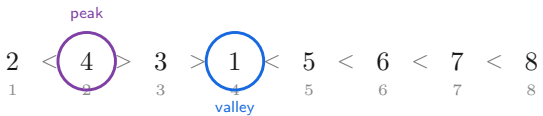

Oğuz's marked-permutation argument gives $d(D; n) = \sum_{J \subseteq \mathrm{Spike}(D)} p_J(n)$. Erasing the marks leaves peak set J; the number of markings is canceled by the power of two in the normalization of p_J.

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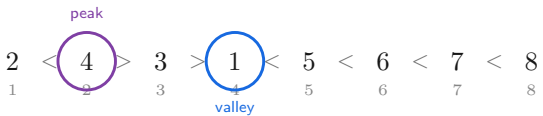

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Example: the four subsets of $\{2, 4\}$ give

$$d(\{2, 3\}; n) = p_{\{2,4\}}(n) + p_{\{2\}}(n) + p_{\{4\}}(n) + p_{\varnothing}(n).$$

Inclusion-Exclusion => Coefficient combinatorics

The peak story now points to one clean open problem

For every admissible S, with $m = \max S$, they prove

$$p_S(n) = \sum_{j=0}^{m-1} (\Delta^j p_S)(m) \binom{n-m}{j}, \qquad (\Delta^j p_S)(m) \geq 0.$$

Development	What it contributes
Oğuz	unflippable representatives for the positive coefficients
Gaetz–Gao	inversion-weighted q-peak polynomials are palindromic
Liang–Sagan	the evaluations $(p_S(n))_n$ are log-concave

OPEN **Díaz-Lopez et al. (2017)**

Must the positive Newton coefficients $\big((\Delta^j p_S)(m)\big)_j$ be log-concave?

Source: Díaz-Lopez–Harris–Insko–Omar, after Theorem 1.2.

Root information can imply log-concavity in favorable settings; the question here is to explain the coefficients themselves.

ACT III: GRAPH PEAKS

An Interlude

Positions become vertices; deletion exposes topology, independence, and order.

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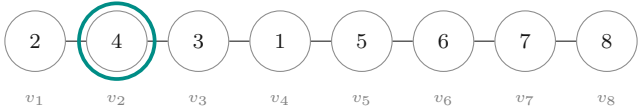
A permutation peak is a local maximum on a labeled path

Put $\sigma = \mathbf{24315678}$ on the path P_8.

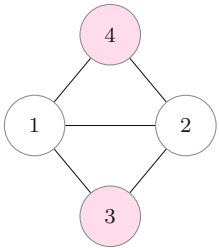


The peak at position 2 is exactly the non-leaf local maximum at v_2.

On a graph, prescribe the vertices that must be local maxima

For a bijective labeling $\lambda : V(G) \to [|V(G)|]$, a graph peak is a vertex v such that

$$\deg(v) \geq 2 \qquad \text{and} \qquad \lambda(v) > \lambda(w) \text{ for every neighbor } w.$$



QUESTION: If $S \subset |V(G)|$, how many labelings λ have peak set exactly S?

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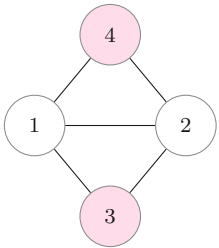


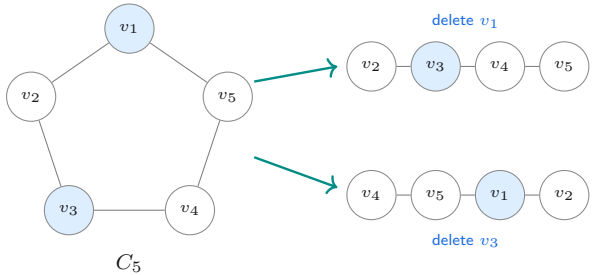
QUESTION: If $S \subset |V(G)|$, how many labelings λ have peak set exactly S?

Admissibility. A nonempty S occurs exactly when:

1. no two vertices in S are adjacent;
2. every $v \in S$ has degree at least 2.

Díaz-Lopez–Everham–Harris–Insko–Marcantonio–Omar, *Counting peaks on graphs* (2019).

On a cycle, maximum deletion returns us to paths



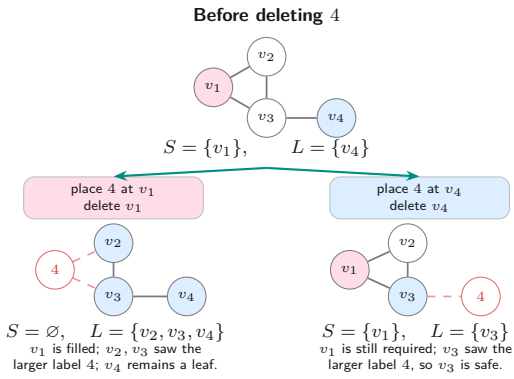
Maximum deletion on C_5; schematic after Díaz-Lopez et al., Fig. 2.1 (2019).

The three moves

1. place n at a shaded vertex;
2. delete that vertex;
3. count on the remaining path.

For C_5, the first choice is v_1 or v_3.

One deletion creates the first split of a decision tree



S = REQUIRED peaks not yet filled.

L = SAFE leaves, or vertices with a larger deleted neighbor.

MAXIMUM-LABEL RULE

Choose

$$v \in S \cup \big(L \setminus N_G(S)\big),$$

label it with the current maximum, and delete it.

Its neighbors now enter L: each has seen a larger label.

Repeat with $3, 2, 1$: each root-to-leaf branch gives one labeling.

Peaks and pinnacles on graphs: the literature at a glance

Authors and work	What is recorded?	What the work established
Burdzy–Pal *Twin peaks* (2016)	One or two local maxima under a uniformly random graph labeling.	Conditional geometry and probability for squares, ladders, and trees.
Davis–Nelson–Petersen–Tenner *The pinnacle set of a permutation* (2018)	The values carried by peaks of a permutation—the path case.	Introduced pinnacle sets, characterized admissibility, and began their enumeration.
Bozeman–Cheng–Harris–Lasinis–Walker *The pinnacle sets of a graph* (2024)	The exact set $P \subseteq [n]$ of local-maximum labels on G; leaves may contribute.	Independent-set criteria and NP-completeness; classifications for $K_n, K_{a,b}, C_n, P_n$; raising moves and a join-semilattice.

Burdzy–Pal (2016); Davis–Nelson–Petersen–Tenner (2018); Díaz-Lopez et al. (2019); Bozeman et al., arXiv:2406.19562 (2024).

Two questions asked in the graph papers

OPEN **Most probable graph peak set**

For fixed G, which admissible S maximizes the number of labelings with peak set S?

Source: Díaz-Lopez et al., *Counting peaks on graphs*, Section 4.

OPEN **Enumeration Question Persists**

Given G and S, how many labelings of G have peak set S?

Source: Díaz-Lopez et al., *Counting peaks on graphs*, Section 4.

ACT IV: DESCENT AND PEAK WORDS

Let the record grow

When new indices keep arriving, polynomials give way to growth rates.

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Until now I or S was fixed: after its largest index, no new prescribed event could occur. An infinite word lets one new local instruction arrive at every n.

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Until now I or S was fixed: after its largest index, no new prescribed event could occur. An infinite word lets one new local instruction arrive at every n.

$$\sigma = \underset{1}{2} < \underset{2}{4} > \underset{3}{3} > \underset{4}{1} < \underset{5}{5} < \underset{6}{6} < \underset{7}{7} < \underset{8}{8}$$

	1	2	3	4	5	6	7
descent word	0	1	1	0	0	0	0
peak word	0	1	0	0	0	0	0

The local profile has become an object of combinatorics on words.

A growth rate measures the exponential cost of a word

For an infinite binary word $w = w_1 w_2 \cdots$:

- $w_1 \cdots w_{n-1}$ prescribes all $n-1$ adjacent comparisons;

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- $r_n(w) = \left(d_n(w)/n!\right)^{1/n}$ isolates the exponential scale.

Study the limit, and when it fails, its two endpoints:

$$\operatorname{gr} d(w) = \lim_{n\to\infty} r_n(w), \qquad \underline{\operatorname{gr}}\, d(w) = \liminf r_n(w), \qquad \overline{\operatorname{gr}}\, d(w) = \limsup r_n(w).$$

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word w	**permutations counted**	$gr\, d(w)$
$0^\omega = 0000\cdots$	only $12\cdots n$, so $d_n(w) = 1$	0
$(10)^\omega = 1010\cdots$	alternating permutations (Euler numbers)	$2/\pi$

Two explicit words anchor the whole spectrum

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Omar and Troyka proved that every value between these anchors occurs:

$$\{\operatorname{gr} d(w) : \text{the limit exists}\} = [0, 2/\pi].$$

Prescribing the indices realizes every possible exponential cost.

Omar–Troyka, European J. Combin. 131 (2026), 104246.

One word can oscillate between any two prescribed rates

Whenever $0 \le L \le L' \le 2/\pi$, Omar and Troyka construct

$$w = 0^{r_1}(10)^{s_1}0^{r_2}(10)^{s_2}\cdots$$

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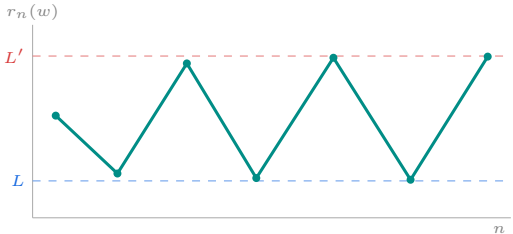
- a long 0^{r_j} block pushes $r_n(w)$ down;
- a long $(10)^{s_j}$ block pushes it up;
- longer blocks make earlier errors negligible.

0^{r_j}**: down** $(10)^{s_j}$**: up**



Local blocks act as a thermostat for a global asymptotic invariant.

Omar–Troyka (2026), constructive algorithm for Theorem 1.2.

Periodic peak spacings give a dense family of rates

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Periodic local rules approximate every allowed exponential scale.

Omar–Troyka (2026).

Two problems from Omar–Troyka

OPEN **Problem 5.1: exact peak spectrum**

Which $L \in [0, 3^{-1/3}]$ occur as actual peak growth rates?

Source: Omar–Troyka, *Growth rates of permutations with given descent or peak set*, Problem 5.1.

OPEN **Problem 5.3: distribution**

What is the distribution of descent growth rates? What is the distribution of peak growth rates?

Source: Omar–Troyka, Problem 5.3.

WHAT APPEARED?

combinatorics on words · asymptotic enumeration · spectra

Both questions invite computation before they demand a new theorem.

ACT V: X-DESCENTS

Change the adjacent rule

Fix the positions and vary which adjacent pairs count: Hamiltonian paths, successions, and transfer matrices appear.

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X-descents keep the positions and change the adjacent rule

Fix a relation $X \subseteq \mathbb{N}^2$ and define

$$\mathrm{XDes}(\sigma) = \{i \in [n-1] : (\sigma(i), \sigma(i+1)) \in X\}.$$

Ordinary descents are recovered from $X = \{(a, b) : a > b\}$.

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Ordinary descents are recovered from $X = \{(a,b) : a > b\}$. For a different example, let

$$X = \{(a,b) : a \text{ is odd and } b \text{ is even}\}.$$

$$\begin{array}{rccccccccccccccc} \sigma = & 2 & < & 4 & > & 3 & > & 1 & < & 5 & < & 6 & < & 7 & < & 8 \\ & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 \\ & & & & & & & & & \underbrace{}_{X} & & & & \underbrace{}_{X} & & \end{array}$$

The underlined adjacent pairs are $5\,6$ and $7\,8$, so

$$\mathrm{XDes}(\sigma) = \{5, 7\}.$$

The positions stay fixed; the meaning of an adjacent event changes.

Omar, *Permutations with a Given X-Descent Set*, arXiv:2402.10443v3 (2026).

No X-descents recover familiar enumerations

The number $d_X(\varnothing; n)$ counts permutations in which $(\sigma_i, \sigma_{i+1}) \notin X$ at every adjacent position i.

Choose X to record	**Then $\mathrm{XDes}(\sigma) = \varnothing$ means**	**Known enumeration or theory**
ordinary descents: $(a,b) \in X$ iff $a > b$	σ is increasing	one permutation: $d_X(\varnothing; n) = 1$
successions: $X = \{(i, i+1) : i \geq 1\}$	no i is immediately followed by $i+1$	inclusion–exclusion: $\displaystyle\sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} (n-j)!$
equal parity: $(a,b) \in X$ iff $a \equiv b \pmod 2$	adjacent entries alternate parity	$\begin{cases} 2((n/2)!)^2, & n \text{ even}, \\ \lfloor n/2 \rfloor! \, \lceil n/2 \rceil!, & n \text{ odd} \end{cases}$
forbidden arcs of a tournament T	$\sigma_1 \to \sigma_2 \to \cdots \to \sigma_n$ is a Hamiltonian path in T	Rédei–Berge: the count is positive and odd; Grinberg–Stanley give a positive formula

$d_X(\varnothing; n)$ is not merely a base case: it unifies several classical enumerations.

Omar, *Permutations with a Given X-Descent Set*, Sections 3.1–3.3; Grinberg–Stanley, arXiv:2307.05569.

Removing the last prescribed X-descent

Assume the count depends only on the *number* of labels chosen, not on the labels themselves. Fix $I \neq \varnothing$ and write

$$m = \max(I), \qquad I^- = I \setminus \{m\}.$$

Cut a permutation immediately after position m:

$$\underbrace{\sigma(1)\cdots\sigma(m)}_{X\text{-descent set } I^-} \;\Big|\; \underbrace{\sigma(m+1)\cdots\sigma(n)}_{\text{no } X\text{-descents}}.$$

boundary m

$$d_X(I;n) = \binom{n}{m} d_X(I^-;m)\, d_X(\varnothing;n-m) - d_X(I^-;n)$$

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Build the two blocks.

- Choose the m labels in the first block.
- Arrange them with X-descent set I^-.
- Arrange the remaining labels with no X-descents.

Correct the boundary.
The product also counts the permutations in which position m is *not* an X-descent. Those have X-descent set I^-, so subtract $d_X(I^-;n)$.

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OPEN **Omar, Question 4.1**

Which verifiable conditions on X make $d_X(I;n)$ eventually polynomial or quasipolynomial for every fixed I?

Source: Omar, *Permutations with a Given X-Descent Set*, Question 4.1.

Having no X-descents is a Hamiltonian-path condition

Build a digraph $G_n(X)$ on $[n]$ by

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$$2 \longrightarrow 4 \longrightarrow 3 \longrightarrow 1.$$

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Thus $d_X(\varnothing; n)$ counts the directed Hamiltonian paths in $G_n(X)$.

Omar, *Permutations with a Given X-Descent Set*, Proposition 3.1.

The same local question keeps opening new subjects

What is prescribed?	What is counted?	What structure appears?
A finite descent set I.	Permutations with $\mathrm{Des}(\sigma) = I$.	Positive bases, posets, tableaux, and q-weights.
A finite peak set S.	Permutations with $\mathrm{Peak}(\sigma) = S$.	Finite differences, insertion, flips, and lattices.
Peak vertices or pinnacle labels on G.	Bijective labelings of the graph.	Deletion, independent sets, complexity, and semilattices.
An infinite descent or peak word.	Permutations following longer prefixes.	Growth-rate spectra and asymptotic control.
An arbitrary adjacent relation X.	Permutations with a fixed X-descent set.	Directed Hamiltonian paths and eventual-polynomiality questions.

The recurring mechanism is

prescribe local sites $\Longrightarrow$ **find the constrained structure** $\Longrightarrow$ **discover a global theory**

Fixed-index avoidance is a meeting place for many parts of combinatorics.

THANK YOU!