

A stretched exponential for $\text{Av}((12, 12), (231, 312)).$

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Joint work with Juliette Schabanel and Paul Thévenin

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PATTERN AVOIDANCE IN 3-PERMUTATIONS

Definition: A *3-permutation* is a pair of permutations of the same length e.g., $(13452, 32541)$

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In 3D: Positions are x and y coordinates, numbers are z -coordinates.
Then 3-permutation $\equiv n$ points in 3D space.

SYMMETRIES OF 3-PERMUTATIONS

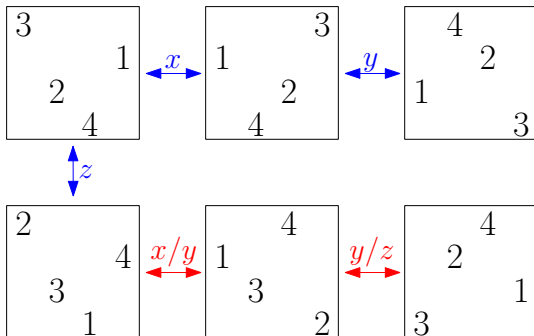
x-complement: Flip in vertical axis

y-complement: Flip in horizontal axis

z-complement: Replace numbers $k \rightarrow n + 1 - k$

x/y switch: Flip in diagonal

y/z switch: exchange heights for numbers



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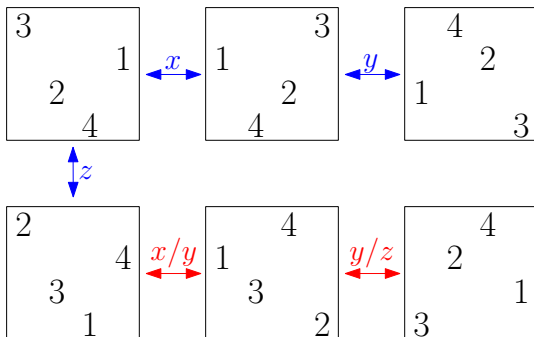
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Potentially 48 symmetric permutations (symmetries of cube).

BACKGROUND ON PATTERN AVOIDING 3-PERMUTATIONS

- [Atkinson, Linton, Walker, 1995] using [Atkinson, Beals, 1994]:

$$\left| \text{Av}_n \left(\begin{array}{|c|c|} \hline & 2 \\ \hline 1 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & \\ \hline & 3 \\ \hline & 2 \\ \hline \end{array} \right) \right| = (n+1)^{n-1}$$

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- [Aldred et al, 2005]: Only two remaining classes (up to symmetries) avoiding a pattern of size 2 and a pattern of size 3:

$$\text{Av} \left(\begin{bmatrix} & 2 \\ 1 & \end{bmatrix}, \begin{bmatrix} & 1 \\ 3 & \\ & & 2 \end{bmatrix} \right) \quad \text{and} \quad \text{Av} \left(\begin{bmatrix} & 2 \\ 1 & \end{bmatrix}, \begin{bmatrix} & & 1 \\ & 2 & \\ 3 & & \end{bmatrix} \right)$$

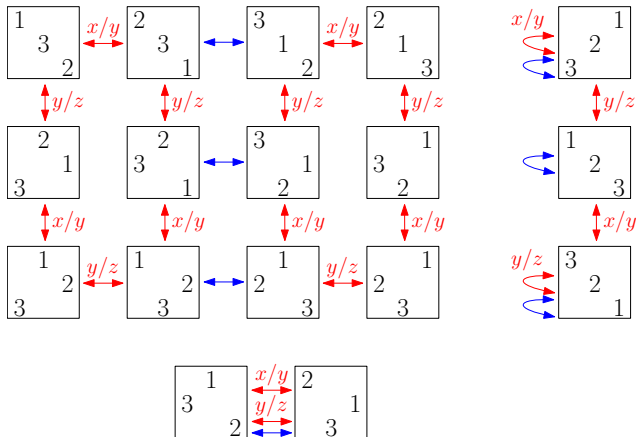
SYMMETRIES OF PATTERNS OF LENGTH 3

Symmetries fixing $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

x/y switch: Flip in diagonal

y/z switch: exchange heights for numbers

complement: rotate 180° and replace numbers $k \rightarrow n + 1 - k$



BACKGROUND ON PATTERN AVOIDING 3-PERMUTATIONS

- [Gunby, Pálvölgy, 2009]: for any 3-permutation π , there are $c_2 > c_1 > 0$ such that as $n \rightarrow \infty$

$$n!^{3/2} c_2^n > |\text{Av}_n(\pi)| > n!^{3/2} c_1^n.$$

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- [Bonichon, Morel, 2022]:

- Bijection: $\text{Av} \left(\begin{array}{|c|c|} \hline & 2 \\ \hline 1 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline & 1 \\ \hline 3 & 2 \\ \hline \end{array} \right)$ equivalent to *intervals in the weak-Bruhat poset that are distributive lattices* (A190291)
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- This work [E.P., Schabanel, Thévenin]:

$$\left| \text{Av}_n \left(\begin{array}{|c|c|} \hline & 2 \\ \hline 1 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline & 1 \\ \hline 3 & 2 \\ \hline \end{array} \right) \right| \sim cn! e^{\sqrt{12}n} n^{5/12},$$

for some $c > 0$.

OUTLINE OF TALK

Define $t_n := \left| \text{Av}_n \left(\begin{array}{|c|} \hline \begin{array}{c} 2 \\ 1 \end{array} \\ \hline \end{array}, \begin{array}{|c|} \hline \begin{array}{c} 1 \\ 2 \end{array} \\ \hline \end{array} \right) \right|$ and $T(z) = \sum_{n=0}^{\infty} t_n z^n$

Outline of talk:

- Part 1: Recurrence for $t_n \equiv$ non-linear differential equation for $T(z)$.
- Part 2: Vague explanation of asymptotic form.

DIFFERENTIAL EQUATION PREVIEW

Recall: $t_n := \left| \text{Av}_n \left(\boxed{\begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix}}, \boxed{\begin{smallmatrix} 1 \\ & 2 \end{smallmatrix}} \right) \right|$ and $T(z) = \sum_{n=0}^{\infty} t_n z^n$.

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Initial terms:

$$T(z) = 1 + t + 3t^2 + 16t^3 + 122t^4 + 1188t^5 + 13844t^6 + \dots$$

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$$I(z) := \frac{T(z) - 1}{\Delta T(z)} = \frac{T(z) - 1}{(zT(z))'}$$

is related to $T(z)$ by

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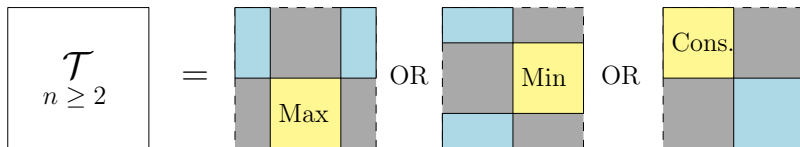
$$T(z) = 1 + z + 3I(z)(\Delta T(z) - 1) - 3I(z)^2 \Delta^2 T(z) + I(z)^3 \Delta^3 T(z).$$

Polynomial time algorithm: We have computed t_n for $n \leq 1000$ (previously there were 16 terms in OEIS).

RECURSIVE DECOMPOSITION

Denote $\mathcal{T} := \text{Av} \left(\begin{smallmatrix} & 2 \\ 1 & \end{smallmatrix}, \begin{smallmatrix} 1 \\ 3 & \\ & 2 \end{smallmatrix} \right)$

Decomposition ([Schabanel, 2025]): Any 3-permutation in $\mathcal{T}$ decomposes in (at least) one of 3 ways:



Max: Contains largest numbers

Min: Contains smallest numbers

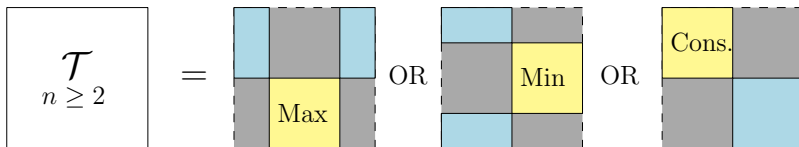
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Colours: Blue and yellow each non-empty

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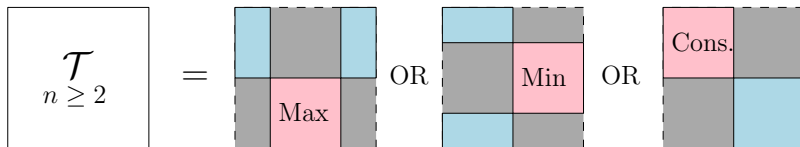
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This work (part 1): Decomposition $\rightarrow$ recurrence for $t_n \equiv$

Differential equation for $T(z)$

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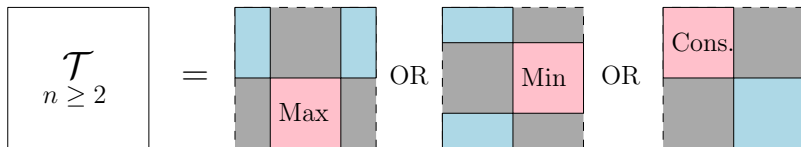
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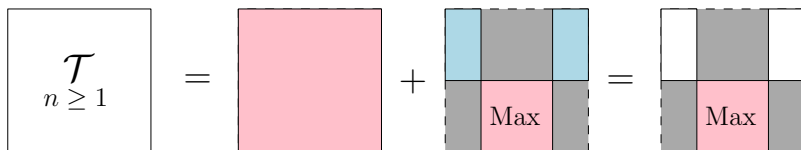
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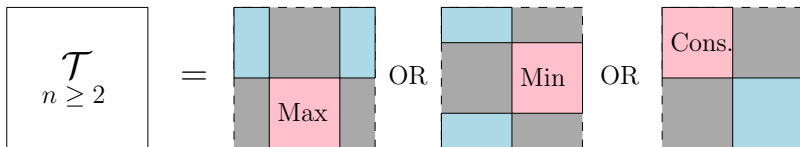
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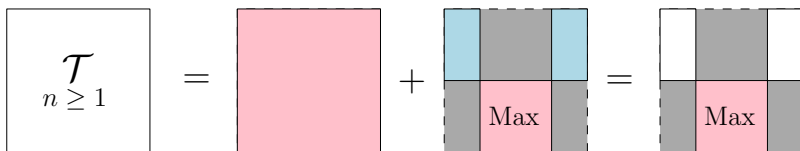
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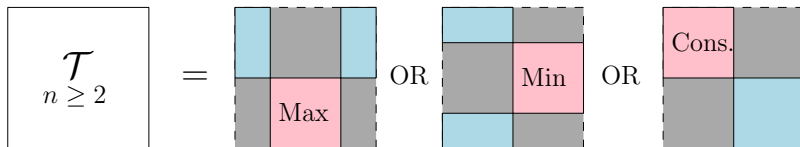
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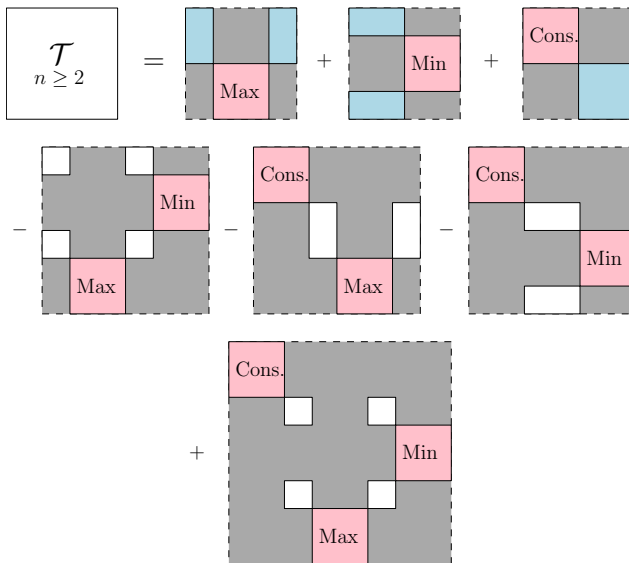


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Next step: Inclusion/exclusion.

$$\begin{aligned}
 \mathcal{T}_{n \geq 2} &= \begin{array}{|c|c|c|} \hline \text{light blue} & \text{grey} & \text{light blue} \\ \hline \text{grey} & \text{Max} & \text{grey} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{light blue} & \text{grey} \\ \hline \text{grey} & \text{Min} \\ \hline \text{light blue} & \text{grey} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \text{Cons.} & \text{grey} \\ \hline \text{grey} & \text{light blue} \\ \hline \end{array} \\
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SUMMARY OF EXACT RESULTS

Recall:

$$t_n := \left| \text{Av}_n \left(\begin{array}{|c|c|} \hline & 2 \\ \hline 1 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline & 1 \\ \hline 3 & \\ \hline & 2 \\ \hline \end{array} \right) \right|,$$

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Next part: asymptotics: $\left| \text{Av}_n \left(\begin{array}{|c|c|} \hline & 2 \\ \hline 1 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline & 1 \\ \hline 3 & \\ \hline & 2 \\ \hline \end{array} \right) \right| \sim cn! e^{\sqrt{12n}} n^{5/12}.$

OUTLINE FOR ASYMPTOTICS

Simplified version: Imagine indecomposable part always has size 1 in inclusion exclusion:

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Matrix form:

$$\begin{pmatrix} \tilde{t}_{n-2} \\ \tilde{t}_{n-1} \\ \tilde{t}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{(n-2)^2}{n(n-1)} & -3\frac{n-1}{n} & 3 \end{pmatrix} \begin{pmatrix} \tilde{t}_{n-3} \\ \tilde{t}_{n-2} \\ \tilde{t}_{n-1} \end{pmatrix}$$

$$\frac{\tilde{t}_n}{\tilde{t}_{n-1}} \approx \max \text{ Eigenvalue of matrix} \approx 1 + \sqrt{\frac{3}{n}}$$

$$\implies \tilde{t}_n \approx e^{\sqrt{12n}} \implies \hat{t}_n \approx n!e^{\sqrt{12n}}.$$

FINAL THOUGHTS

First class: OEIS A000272:

$$\left| \text{Av}_n \left(\boxed{1 \atop 2}, \boxed{1 \atop 3 \atop 2} \right) \right| = (n+1)^{n-1} \sim cn!e^n n^{-3/2},$$

Exponential generating function D-algebraic.

Second class: OEIS A295928:

$$\left| \text{Av}_n \left(\boxed{1 \atop 2}, \boxed{3 \atop 1 \atop 2} \right) \right| \sim cn!e^{\sqrt{12n}} n^{5/12},$$

Ordinary generating function D-algebraic.

Third class: OEIS A190291: $\left| \text{Av}_n \left(\boxed{1 \atop 2}, \boxed{3 \atop 2 \atop 1} \right) \right| ??$

FINAL THOUGHTS

First class: OEIS A000272:

$$\left| \text{Av}_n \left(\boxed{1 \atop 2}, \boxed{1 \atop 3 \atop 2} \right) \right| = (n+1)^{n-1} \sim cn!e^n n^{-3/2},$$

Exponential generating function D-algebraic.

Second class: OEIS A295928:

$$\left| \text{Av}_n \left(\boxed{1 \atop 2}, \boxed{3 \atop 1 \atop 2} \right) \right| \sim cn!e^{\sqrt{12n}} n^{5/12},$$

Ordinary generating function D-algebraic.

Third class: OEIS A190291: $\left| \text{Av}_n \left(\boxed{1 \atop 2}, \boxed{3 \atop 2 \atop 1} \right) \right| ??$

Thank You!!