

Well-nested permutations and signed permutations

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Outline

This talk is about certain (signed) permutations w characterized in the following way:

- (a) Start with an equivalence relation and consider the class containing w .
- (b) Insist that every element of this class avoid certain (consecutive) patterns.

The relevant equivalence class may be very large, making it difficult to check (b) directly.

However, in our cases, there will turn out to be much more efficient ways to verify this.

These cases will be closely connected to both

- (1) the geometry of complete flag varieties, and
- (2) a natural problem in Coxeter combinatorics related to Demazure products.

Well-nested permutations

Throughout let $\sim$ be the equivalence relation on partial permutations generated by

$$\dots BCA \dots \sim \dots CAB \dots \quad \text{whenever } A < B < C$$

Definition

A partial permutation is **well-nested** if no element in its equivalence class under $\sim$ has any **consecutive 321-patterns**.

A **well-nested family** is a union of such equivalence classes.

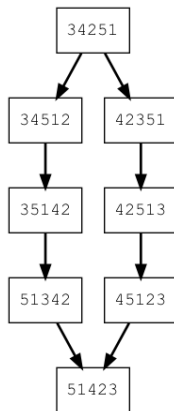
The elements of a well-nested family are all well-nested.

Example

The set $E = \{34251, 34512, 35142, 42351, 42513, 45123, 51342, 51423\}$ is a well-nested family which is **minimal**, i.e., a single equivalence class under $\sim$.

Well-nested family as digraphs

We may draw the preceding example as a directed graph



where edges are the covering relations $\cdots BCA \cdots \prec \cdots CAB \cdots$ (when $A < B < C$).

Well-nested structure

Well-nested families have a surprisingly rigid structure and a complete classification.

Theorem (Hamaker–M.–Pawlowski)

(a) Any well-nested family of permutations is a graded poset under the transitive relation with

$$\dots BCA \dots \prec \dots CAB \dots \quad (\text{when } A < B < C)$$

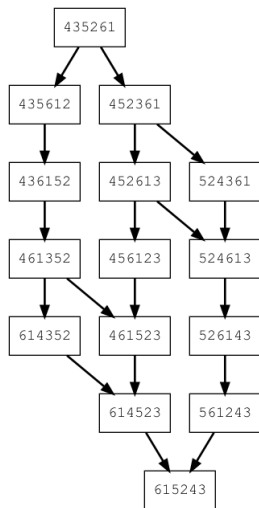
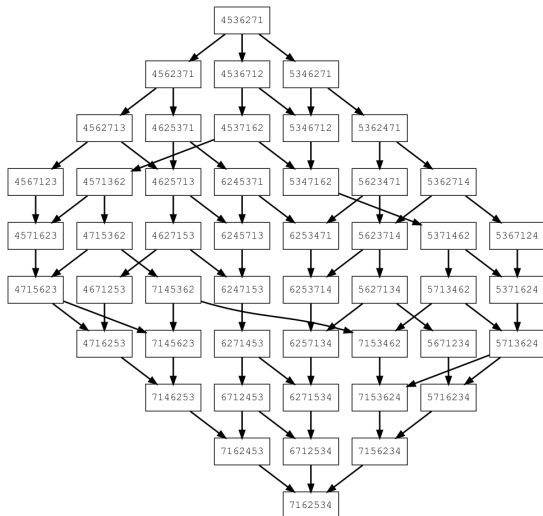
and any interval in this poset has a unique minimal element and a unique maximal element.

(b) The minimal well-nested families in S_n are in bijection with $\text{Invol}(S_n) = \{z = z^{-1} \in S_n\}$.

Conjecture

Any minimal well-nested family is a lattice under $\prec$.

Two more minimal well-nested families



Nested descent sets

Assume w is a partial permutation of length n . An index i is a **descent** of w if $w_i > w_{i+1}$. Define the **nested descent set** and **nested residue set** of w recursively by

$$\text{NDes}(w) = \begin{cases} \emptyset & \text{if } w \text{ has no descents} \\ \{(w_i, w_{i+1})\} \sqcup \text{NDes}(w_1 \cdots w_{i-1} w_{i+2} \cdots w_n) & \text{if } i \text{ is the first descent} \end{cases}$$

$$\text{NRes}(w) = \begin{cases} \{w_1, w_2, \dots, w_n\} & \text{if } w \text{ has no descents} \\ \text{NRes}(w_1 \cdots w_{i-1} w_{i+2} \cdots w_n) & \text{if } i \text{ is the first descent.} \end{cases}$$

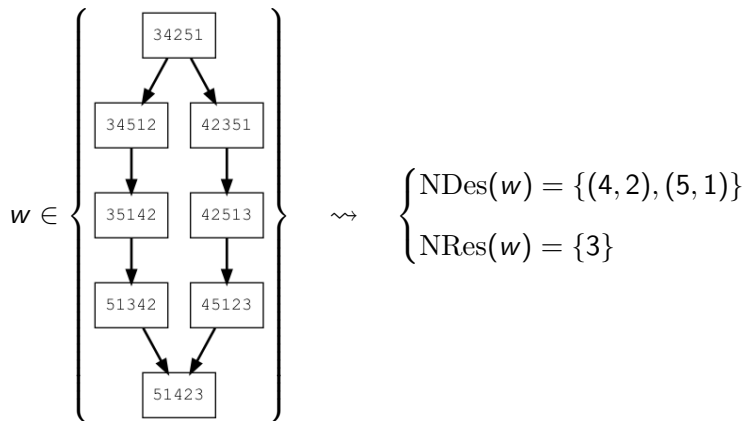
Theorem (M.)

If v and w are both well-nested then $v \sim w$ if and only if

$$\text{NDes}(v) = \text{NDes}(w) \quad \text{and} \quad \text{NRes}(v) = \text{NRes}(w).$$

In this case “first descent” can be replaced by “any descent” in above definitions.

Examples



Define $\boxed{\text{invol}_A(w) \in \text{Invol}(S_n)}$ to have 2-cycles $\text{NDes}(w)$, so $\text{invol}_A(34512) = (1, 5)(2, 4)$. Then $w \mapsto \text{invol}_A(w)$ is the bijection from minimal well-nested families in S_n to $\text{Invol}(S_n)$.

Testing well-nestedness

Among all minimal well-nested families in S_n , the largest has size $(n-1)!!$.

All well-nested permutations w in this family have $\text{invol}_A(w) = n \cdots 321$.

Whether or not a partial permutation is well-nested can be checked more efficiently:

Theorem (Can–Joyce–Wyser)

A partial permutation w is well-nested if and only whenever

$$(b_1, a_1), (b_2, a_2) \in \text{NCyc}(w) := \text{NDes}(w) \sqcup \{(c, c) : c \in \text{NRes}(w)\}$$

have $a_1 < a_2$ and $b_1 < b_2$, the number a_1 appears to the left of b_2 in w .

$$6245713 \rightsquigarrow \text{NCyc} = \{(6, 2), (7, 1), (5, 3), (4, 4)\} \rightsquigarrow \checkmark$$

$$6415273 \rightsquigarrow \text{NCyc} = \{(6, 4), (7, 3), (5, 2), (1, 1)\} \rightsquigarrow \times$$

$$6245731 \rightsquigarrow \text{NCyc} = \{(6, 2), (7, 3), (5, 1), (4, 4)\} \rightsquigarrow \times$$

Signed permutations

A **signed permutation** is a partial permutation $w = w_1 w_2 \cdots w_n$ with $|w_1| |w_2| \cdots |w_n| \in S_n$.

Let W_n^B be the set of signed permutations of length n . Write $\bar{x}$ instead of $-x$, so

$$W_2^B = \{12, 21, \bar{1}2, 2\bar{1}, 1\bar{2}, \bar{2}1, \bar{1}\bar{2}, \bar{2}\bar{1}\}.$$

Definition

A signed permutation is **absolutely well-nested** if no element in its equivalence class under

$$\cdots BCA \cdots \sim \cdots CAB \cdots \quad \text{for } A < B < C$$

has any **consecutive 321-patterns** or any **consecutive $\bar{1}2$ - or $1\bar{3}\bar{2}$ -patterns**

Call any union of equivalence classes of these elements an **absolutely well-nested family**.

Having a consecutive $1\bar{3}\bar{2}$ -pattern, for example, means we can find $0 < w_i < -w_{i+2} < -w_{i+1}$.

Shapes and involutions

Given $w \in W_n^B$ define its **shape** to be the matching

$$\text{sh}_B(w) = \{\pm\{a, -b\} : (a, b) \in \text{NDes}(w) : a < -b\} \sqcup \{\{-c, c\} : 0 > c \in \text{NRes}(w)\}.$$

Example

For $w = 1\bar{4}6\bar{5}37\bar{8}2$: $\text{NDes}(w) = \{(1, \bar{4}), (6, \bar{5}), (7, \bar{8}), (3, 2)\}$ so $\text{sh}_B(w) = \{\bar{8}\bar{7} \mid \bar{4}\bar{1} \mid 14 \mid 78\}$.

Can view W_n^B as group of bijections $w : [\pm n] \rightarrow [\pm n] = \{\pm 1, \dots, \pm n\}$ with $w(-i) = -w(i)$.

Write $S \subseteq [\pm n]$ for union of blocks in $\text{sh}_B(w)$. Let $\boxed{\text{invol}_B(w) \in \text{Invol}(W_n^B)}$ be product of $(a, b)(-a, -b)$ for $(a, b) \in \text{NDes}(w)$ with $a > -b$, and $(-c, c)$ for $c \in S$.

Example

$$\text{invol}_B(1\bar{4}6\bar{5}37\bar{8}2) = (-1, 1)(2, 3)(-2, -3)(-4, 4)(5, -6)(6, -5)(-7, 7)(-8, 8) = \bar{1}32\bar{4}\bar{6}\bar{5}\bar{7}\bar{8}.$$

Characterizing absolutely well-nested permutations

Theorem (Hamaker–M.)

A signed permutation $w \in W_n^B$ is absolutely well-nested if and only if

- (a) w is well-nested as a partial permutation, and
- (b) $\text{sh}_B(w)$ is a noncrossing matching.

Theorem (Hamaker–M.)

The map $w \mapsto (\text{invol}_B(w), \text{sh}_B(w))$ is constant on $\sim$ -equivalence classes, and descends to a bijection from minimal absolutely well-nested families in W_n^B to pairs (z, M) where

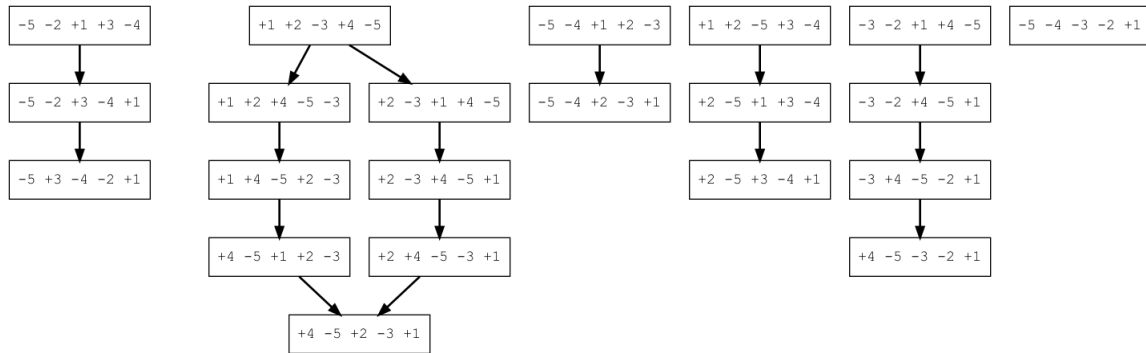
$z \in \text{Invol}(W_n^B)$, and

M is a symmetric, perfect, noncrossing matching on $\{i \in [\pm n] : z(i) = -i\}$.

Here **symmetric** means invariant under negation, **perfect** means every block has size two, and **noncrossing** means that no blocks $\{a_1 < b_1\}$ and $\{a_2 < b_2\}$ have $a_1 < a_2 < b_1 < b_2$.

Signed examples

Absolutely well-nested $w \in W_5^B$ with $\text{invol}_B(w) = 1\bar{2}\bar{3}\bar{4}\bar{5}$, grouped into minimal families:



We have $\text{sh}_B(\bar{5}\bar{2}13\bar{4}) = \{ \bar{5}\bar{5} \mid \bar{1}\bar{1} \mid \bar{4}\bar{3} \mid 34 \}$ and $\text{sh}_B(12\bar{3}\bar{4}\bar{5}) = \{ \bar{5}\bar{4} \mid \bar{3}\bar{2} \mid 23 \mid 45 \}$.

Even-signed permutations

Let $W_n^D \subset W_n^B$ be the subgroup of **even-signed permutations**, meaning a signed permutation $w = w_1 w_2 \cdots w_n$ with an **even number of negative letters**:

$$W_2^D = \{12, 21, \bar{1}\bar{2}, \bar{2}\bar{1}\}.$$

Define $\tilde{w} = \bar{w}_1 w_2 w_3 \cdots w_n$. For each $w \in W_n^B$ exactly one of w or $\tilde{w}$ is in W_n^D .

Let $\prec_D$ be the transitive relation on W_n^D with $v \prec_D w$ if $v \prec w$ or $\tilde{v} \prec \tilde{w}$ where as usual

$$\cdots BCA \cdots \prec \cdots CAB \cdots \quad (\text{when } A < B < C)$$

Write $\sim^D$ for the reflexive, symmetric closure of this relation.

Definition

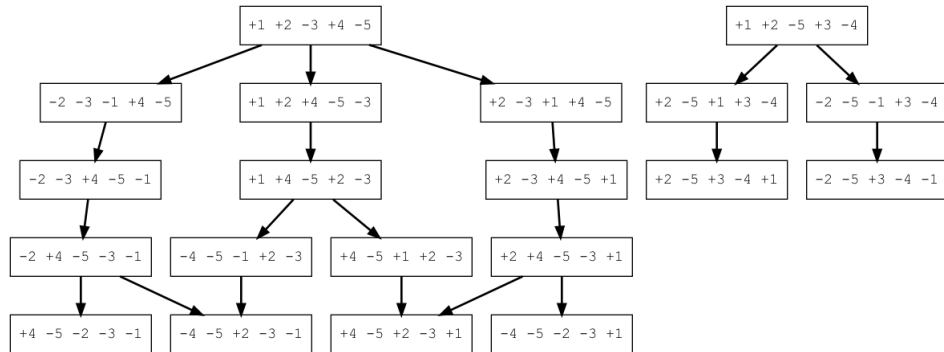
An even-signed partial permutation is **evenly well-nested** if no element in its equivalence class under $\sim^D$ has any consecutive 321-patterns or consecutive $1\bar{3}\bar{2}$ -patterns.

An **evenly well-nested family** is a union of evenly well-nested $\sim^D$ -equivalence classes.

Evenly well-nested structures

Theorem (M.)

Any evenly well-nested family in W_n^D is a graded poset under $\prec_D$ with a unique minimum in each lower interval. (But an upper interval may have multiple maxima.)



Nested descent sets for even signed permutations

Fix $w \in W_n^D$ and write $\{0\} \sqcup \text{NRes}(w) = \{a_1 < \cdots < a_p < 0 < a_{p+1} < \cdots < a_{p+q}\}$.

Then define $\text{NDes}^\pm(w)$ to consist of
$$\begin{cases} (|a|, b) & \text{for each } (a, b) \in \text{NDes}(w); \\ (|a_i|, a_{i+1}) & \text{for each odd } i \in [p-1]; \text{ and} \\ (|a_p|, a_{p+1}) & \text{if } p \text{ is odd and } q > 0 \text{ and } |a_p| > a_{p+1}. \end{cases}$$

Let $\text{NRes}^\pm(w) = \begin{cases} \{a_{p+2}, \dots, a_{p+q}\} & \text{if } p \text{ is odd and } q > 0 \text{ and } |a_p| > a_{p+1} \\ \{|a_p|, a_{p+1}, \dots, a_{p+q}\} & \text{if } p \text{ is odd and } q > 0 \text{ and } |a_p| < a_{p+1} \\ \{a_{p+1}, \dots, a_{p+q}\} & \text{otherwise.} \end{cases}$

Theorem (M.)

If $v, w \in W_n^D$ are both evenly well-nested then $v \stackrel{D}{\sim} w$ if and only if

$$\text{NDes}^\pm(v) = \text{NDes}^\pm(w) \quad \text{and} \quad \text{NRes}^\pm(v) = \text{NRes}^\pm(w).$$

Shapes and involutions in type D

Given $w \in W_n^D$ define its **shape** to be the matching

$$\text{sh}_D(w) = \{ \{|a|, -b\} : (a, b) \in \text{NDes}(w) : |a| < -b \}.$$

Write $S \subseteq [\pm n]$ for union of blocks in $\text{sh}_D(w)$. Let $\boxed{\text{invol}_D(w) \in \text{Invol}(W_n^D)}$ be product of $(a, b)(-a, -b)$ for $(a, b) \in \text{NDes}^\pm(w)$ with $|a| > -b$, and $(-c, c)$ for $c \in S$.

Example

If $w = \bar{6}\bar{5}\bar{1}\bar{2}3487 \in W_8^D$ then

$$\text{NDes}(w) = \{(\bar{1}, \bar{2}), (8, 7)\}$$

$$\text{NRes}(w) = \{\bar{6}, \bar{5}, 3, 4\}$$

$$\text{NDes}^\pm(w) = \{(1, \bar{2}), (8, 7), (6, \bar{5})\}$$

$$\text{NRes}^\pm(w) = \{3, 4\}$$

so

$$\text{sh}_D(w) = \{12\}$$

$$\text{invol}_D(w) = (\bar{1}, 1)(\bar{2}, 2)(6, \bar{5})(5, \bar{6})(7, 8)(\bar{7}, \bar{8}).$$

Characterizing evenly well-nested permutations

As with S_n and W_n^B , there is a fast algorithm to determine if $w \in W_n^D$ is evenly well-nested.

The precise characterization is a little technical. Some necessary conditions:

- (a) w and $\tilde{w}$ are well-nested as partial permutations, and
- (b) $\text{sh}_D(w)$ is a noncrossing matching.

Theorem (M.)

The map $w \mapsto (\text{invol}_D(w), \text{sh}_D(w))$ is constant on $\sim^D$ -equivalence classes, and descends to a bijection from minimal evenly well-nested families in W_n^D to pairs (z, M) where

$z \in \text{Invol}(W_n^D)$, and

M is a perfect noncrossing matching on $\{i \in [n] : z(i) = -i\}$.

Demazure products

Let W be a Coxeter group with simple generating set S and length function $\ell : W \rightarrow \mathbb{N}$. Then $\ell(w)$ is the minimal number of factors in any product of simple generators that gives w .

The **Demazure product** is another associative multiplication $\circ$ on W , characterized by

$$v \circ w = vw \text{ if } \ell(vw) = \ell(v) + \ell(w) \quad \text{and} \quad s \circ s = s \text{ if } s \in S.$$

We view S_n , W_n^B , and W_n^D as Coxeter groups with simple generators $s_i = (i, i+1)$ or

$$s_i^B = (i, i+1)(-i, -i-1) \text{ and } s_0^B = (-1, 1) \quad \text{or} \quad s_i^D = s_i^B \text{ and } s_0^D = (1, -2)(2, -1).$$

Theorem (M.)

Assume $W \in \{S_n, W_n^B, W_n^D\}$ and $v \in W$. Then the following are equivalent:

- (a) Every $w \in W$ with $v \circ v^{-1} = w \circ w^{-1}$ has $\ell(v) \leq \ell(w)$.
- (b) The element v is (absolutely/evenly) well-nested.

Coxeter generalizations

The relations $\sim$ and $\stackrel{D}{\sim}$ are special cases of a relation $\sim_W$ on any Coxeter group W .

Namely, define $\sim_W$ to be the transitive closure of the relation on W that has

$$wst \sim_W wts$$

whenever $w \in W$ and $s, t \in S$ are such that

$$sts = tst \quad \text{and} \quad \ell(wst) = \ell(wts) = \ell(w) + 2.$$

Under these conditions

$$(wst) \circ (wst)^{-1} = w \circ s \circ t \circ t \circ s \circ w^{-1} = w \circ s \circ t \circ s \circ w^{-1} = w \circ sts \circ w^{-1}$$

which makes it easy to see that $(wst) \circ (wst)^{-1} = (wts) \circ (wts)^{-1}$.

Connections to geometry

Well-nested families have applications to the geometry of orbit closures in flag varieties.

Let G be an algebraic group defined over $\mathbb{C}$, with a holomorphic involution $\theta \in \text{Aut}(G)$.

Choose a Borel subgroup $B = \theta(B)$.

Then $K = \{g \in G : g = \theta(g)\}$ acts on **complete flag variety** G/B with finitely many orbits.

Definition

The Weyl group W of G indexes a family of **(opposite) Schubert varieties**

$$X_w = \overline{Bw_0w^{-1}B}/B$$

whose cohomology classes are a $\mathbb{Z}$ -basis for the cohomology ring $H^*(G/B)$.

The K -orbits in G/B have a complete classification. A formula of Brion gives the Schubert expansion of the cohomology class of the closures of these orbit.

Brion's formula in balanced classical types

Assume we are in one of the following **balanced** (G, K) types:

	G	K	W
Type A	$\mathrm{GL}_n(\mathbb{C})$	$\mathrm{O}_n(\mathbb{C})$	S_n
Type B	$\mathrm{SO}_{2n+1}(\mathbb{C})$	$\mathrm{S}(\mathrm{O}_n(\mathbb{C}) \times \mathrm{O}_{n+1}(\mathbb{C}))$	W_n^{B}
Type C	$\mathrm{Sp}_{2n}(\mathbb{C})$	$\mathrm{GL}_n(\mathbb{C})$	W_n^{B}
Type D	$\mathrm{SO}_{2n}(\mathbb{C})$	$\mathrm{S}(\mathrm{O}_n(\mathbb{C}) \times \mathrm{O}_n(\mathbb{C}))$	W_n^{D}

Corollary (Brion)

In these cases, the cohomology class of each K -orbit closure $Y \subseteq G/B$ has the form

$$[Y] = \sum_{w \in E} 2^{d(w)} [X_w]$$

where $d : W \rightarrow \mathbb{N}$ is a certain map and E is some (evenly/absolutely) well-nested family in W .

In type A we may always take E to be a minimal family; this does not extend to other types.

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Thanks!