

When do three prefix reversals generate S_n ?

Taking a closer look at cubic pancake graphs

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What is a prefix reversal?

For $2 \leq t \leq n$, the prefix reversal $r_t \in \text{Sym}_n$ reverses the first t positions:

$$r_t(i) = \begin{cases} t+1-i, & i \leq t, \\ i, & i > t. \end{cases} \quad r_t = (1 \ t)(2 \ t-1) \cdots$$



Parity of a flip

Since r_t is a product of $\lfloor t/2 \rfloor$ transpositions, $\text{sgn}(r_t) = (-1)^{\lfloor t/2 \rfloor}$.

Why study triples?

The full pancake graph is

$$P_n = \text{Cay}(\text{Sym}_n, \{r_2, r_3, \dots, r_n\}).$$

Restricting to three flips produces a cubic Cayley graph whenever the flips generate Sym_n .

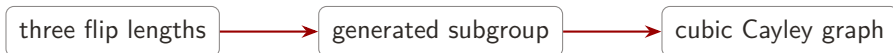
Algebraic question

For which distinct lengths i, j, k do we have

$$\langle r_i, r_j, r_k \rangle = \text{Sym}_n?$$

Graph-theoretic payoff

A generating triple yields a connected cubic pancake network, whose diameter and Hamiltonicity can then be studied.



What is known?

Bass and Sudborough exhibited six infinite generating families; representative examples include

$$\{r_n, r_{n-1}, r_2\}, \quad \{r_n, r_{n-1}, r_{n-2}\}, \quad \{r_n, r_{n-2}, r_3\} \quad (n \text{ even}).$$

Gun later added $\{r_n, r_{n-2}, r_2\}$ for odd n .

Recent progress

Previous work by the authors classified triples containing one of

$$r_2, \quad r_3, \quad r_{n-1}, \quad r_{n-2},$$

and proposed broader conjectures about how far the middle flip can lie from n . Here, we prove some of these conjectures.



Primitive group actions

Let $G \leq S_n$ act naturally on $\Omega = \{1, 2, \dots, n\}$.

Block of imprimitivity

A nonempty subset $B \subseteq \Omega$ is a *block* if, for every $g \in G$,

$$g(B) = B \quad \text{or} \quad g(B) \cap B = \emptyset.$$

The action of G is *primitive* if

- ▶ G acts transitively on Ω ; and
- ▶ the only blocks are the singletons and Ω itself.

Intuition

A primitive action admits no nontrivial way to divide the points into smaller groups that are permuted as whole units by G .

Thinking of the gap between generators

Consider

$$H = \langle r_n, r_{n-\ell}, r_k \rangle, \quad 1 < k < n - \ell < n,$$

1. Orbits

Is H transitive?

2. Blocks

Is H primitive?

3. Size

Does $H \geq \text{Alt}_n$?

4. Parity

Does H contain an odd permutation?

The recurring strategy is: rule out invariant subsets and blocks, force Alt_n , then use parity to obtain Sym_n .

Two basic intransitivity criteria

For $H = \langle r_n, r_{n-\ell}, r_k \rangle$, earlier work supplied two useful obstructions.

Large gap obstruction

For sufficiently large n , if

$$\ell \geq 2k + 1,$$

then $H < \text{Sym}_n$.

Divisibility obstruction

If $n \geq 6$, $\ell \geq k + 1$, and $\ell \mid n + 1$, then

$$\Delta = \{t \in [n] : \ell \mid t\}$$

is a proper H -invariant subset. Hence H is intransitive.

Example I: the triple $\{r_{11}, r_8, r_2\}$

Here

$$n = 11, \quad k = 2, \quad n - \ell = 8, \quad \ell = 3.$$

Since $3 \mid 12 = n + 1$, the set $\Delta = \{3, 6, 9\}$ below is invariant

$$r_2(\Delta) = r_8(\Delta) = r_{11}(\Delta) = \Delta.$$



r_{11} and r_8 permute the highlighted positions; r_2 fixes them.

Thus every reachable permutation has the pattern

$$x \times a \times x \times b \times x \times c \times x, \quad \{a, b, c\} = \{3, 6, 9\}.$$

Therefore Δ is a proper invariant subset and

$$\langle r_{11}, r_8, r_2 \rangle < \text{Sym}_{11}.$$

Example II: the triple $\{r_{18}, r_{13}, r_2\}$

Now $n = 18$, $k = 2$, and $\ell = 5 = 2k + 1$

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18		

positions arranged modulo 5

The set

$$\Delta = \{4, 5, 9, 10, 14, 15\}$$

is preserved by all three flips:

$$r_{18}(\Delta) = \Delta, \quad r_{13}(\Delta) = \Delta, \quad r_2(\Delta) = \Delta.$$

Equivalently, the last two columns form a union of orbits. Hence

$$\langle r_{18}, r_{13}, r_2 \rangle < \text{Sym}_{18}.$$

New Theorem 1: a sharp intransitivity range

Theorem

Let

$$H = \langle r_n, r_{n-\ell}, r_k \rangle, \quad 2 \leq k < n - \ell < n.$$

Then $H < \text{Sym}_n$ whenever either

$$n \text{ even and } \ell \geq k + 1$$

or

$$n \text{ odd and } \ell \geq k + 2.$$



This improves the earlier sufficient obstruction $\ell \geq 2k + 1$ to essentially the best possible linear threshold.

Triangle of misery

Fix n , and consider the parameter triangle

$$2 \leq k < m < n.$$

Each point (m, k) represents the triple

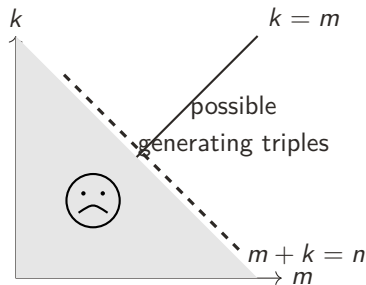
$$\{r_n, r_m, r_k\}.$$

- ▶ A marked point means that

$$r_n, r_m, r_k = \text{Sym}_n.$$

- ▶ A point in the gray region represents a triple that does not generate Sym_n .
- ▶ The relevant boundary is approximately

$$m + k = n.$$



The gray triangular region consists of the points lying “too far below” the boundary $m + k \approx n$. 11/25

Proof Idea I: Arrange $[n]$ in columns modulo ℓ

Write

$$n = q\ell + r, \quad 0 \leq r < \ell,$$

and define

$$C_s = \{s + t\ell : 0 \leq t < q_s\}, \quad q_s = \begin{cases} q + 1, & s \leq r, \\ q, & s > r. \end{cases}$$

For $n = 18$ and $\ell = 5$:

C_1	C_2	C_3	C_4	C_5	
1	2	3	4	5	} step size ℓ
6	7	8	9	10	
11	12	13	14	15	
16	17	18			

The column picture reduces the action to a graph on only ℓ vertices.

How the two large flips move columns

Define the involution

$$\alpha(s) = \begin{cases} r + 1 - s, & s \leq r, \\ \ell + r + 1 - s, & s > r. \end{cases}$$

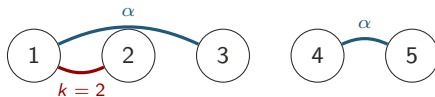
Then both r_n and $r_{n-\ell}$ send the entire column C_s to $C_{\alpha(s)}$ (with different vertical reversals).

For $n = 18$, $\ell = 5$, and $r = 3$,

The small flip r_k touches only the first row and connects

$$\alpha = (1\ 3)(4\ 5), \quad \alpha(2) = 2.$$

$$s \longleftrightarrow k + 1 - s, \quad 1 \leq s \leq k.$$



column-level action

Proof Idea II: The graph $\Gamma(n, \ell, k)$

Definition

The vertex set is $\{1, \dots, \ell\}$. Add

- ▶ an α -edge $\{s, \alpha(s)\}$ whenever $s \neq \alpha(s)$;
- ▶ a k -edge $\{s, k+1-s\}$ for the nonfixed pairs of r_k .

If S is a union of connected components, then

$$U_S = \bigcup_{s \in S} C_s$$

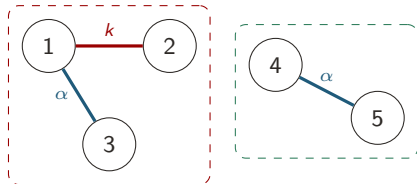
is H -invariant.

Immediate consequence

If $\Gamma(n, \ell, k)$ is disconnected, then H is intransitive and hence $H < \text{Sym}_n$.

Thus a permutation-group problem on n points becomes a connectivity problem on ℓ columns.

Example: $\Gamma(18, 5, 2)$



The components are

$$\{1, 2, 3\} \quad \text{and} \quad \{4, 5\}.$$

The second component gives

$$U_{\{4,5\}} = C_4 \cup C_5 = \{4, 5, 9, 10, 14, 15\},$$

exactly the invariant subset from Example II.

Interpretation

The graph records which residue columns can communicate under the three flips. Different components can never exchange points.

Why $\ell \geq k + 3$ forces disconnection

Every vertex of $\Gamma(n, \ell, k)$ is incident to

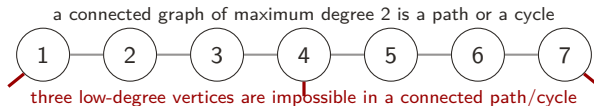
- ▶ at most one α -edge, and
- ▶ at most one k -edge.

Hence every vertex has degree at most 2.

If $\ell \geq k + 3$, then the vertices

$$k + 1, k + 2, \dots, \ell$$

have no k -edge, so at least three vertices have degree at most 1.



Therefore Γ is disconnected and $H < \text{Sym}_n$. This proves the theorem whenever $\ell \geq k + 3$.

Counting admissible middle flips

Fix the small flip k and define

$$F_k(n) = \{m : k < m < n, \langle r_n, r_m, r_k \rangle = \text{Sym}_n\}.$$

Since $m = n - \ell$, New Theorem 1 gives

$$F_k(n) \subseteq \begin{cases} \{n - k, n - k + 1, \dots, n - 1\}, & n \text{ even,} \\ \{n - k - 1, n - k, \dots, n - 1\}, & n \text{ odd.} \end{cases}$$

If k is odd, parity forces m to be even. Therefore

$$\max_{n > k+1} |F_k(n)| \leq \begin{cases} k + 1, & k \text{ even,} \\ \frac{k + 1}{2}, & k \text{ odd.} \end{cases}$$

The next result shows that these upper bounds are attained.

New Theorem 2!: the exact maximum for $F_k(n)$

Theorem

For every $k \geq 2$,

$$\max_{n > k+1} |F_k(n)| = \begin{cases} k+1, & k \text{ even,} \\ \frac{k+1}{2}, & k \text{ odd.} \end{cases}$$

A small sharp example

For $k = 2$ and $n = 7$,

$$F_2(7) = \{4, 5, 6\}.$$

Thus all three triples

$$\{r_7, r_4, r_2\}, \quad \{r_7, r_5, r_2\}, \quad \{r_7, r_6, r_2\}$$

generate Sym_7 , attaining the bound $k+1 = 3$.

Building the matching lower bound

Define

$$\mathcal{L}_k = \begin{cases} \{1, 2, \dots, k+1\}, & k \text{ even,} \\ \{1, 3, 5, \dots, k\}, & k \text{ odd.} \end{cases} \quad |\mathcal{L}_k| = \begin{cases} k+1, & k \text{ even,} \\ (k+1)/2, & k \text{ odd.} \end{cases}$$

Number-theoretic choice of degree

The Chinese remainder theorem and Dirichlet's theorem provide arbitrarily large primes

$$n \equiv 3 \pmod{4}, \quad \gcd(n-k, \ell) = 1 \quad (\ell \in \mathcal{L}_k).$$

For each $\ell \in \mathcal{L}_k$, set $m = n - \ell$ and

$$H_\ell = \langle r_n, r_{n-\ell}, r_k \rangle.$$

It remains to prove that every one of these $|\mathcal{L}_k|$ groups equals Sym_n .

Transitivity from modular shifts

For $i > \ell$,

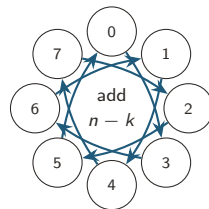
$$r_{n-\ell} r_n(i) = i - \ell.$$

Thus every point moves to the smallest positive representative of its residue class modulo ℓ .

On these representatives, composition with the small flip induces

$$s \mapsto s + (n - k) \pmod{\ell},$$

Since $\gcd(n - k, \ell) = 1$, this translation is a single cycle on the residue classes. Hence H_ℓ is transitive.



From transitivity to Sym_n

The chosen degree n is prime, so transitivity implies primitivity.

Let $\mu(H)$ be the minimum number of points moved by a nonidentity element of H . Since $r_k \in H$,

$$\mu(H) \leq k.$$

Babai's minimum-degree theorem gives, for a primitive group not containing Alt_n ,

$$\mu(H) > \frac{\sqrt{n} - 1}{2}.$$

Choose $n > (2k + 1)^2$. Then the inequality above does not hold, so

$$\text{Alt}_n \leq H_\ell.$$

Finally, $n \equiv 3 \pmod{4}$ makes r_n odd, and therefore

$$H_\ell = \text{Sym}_n.$$

This supplies $|\mathcal{L}_k|$ distinct generating values of m and proves the lower bound.

New Theorem 3!: primitive triples are large

Theorem (Primitive triple theorem)

Let

$$2 < a < b < n, \quad G = \langle r_a, r_b, r_n \rangle.$$

If G is primitive (hence transitive), then

$$G \in \{\text{Alt}_n, \text{Sym}_n\}.$$

Why this matters

Once primitivity is established, there are no intermediate primitive subgroups to consider: the group is already alternating or symmetric.

Final parity step

If one of r_a, r_b, r_n is odd, then the conclusion strengthens immediately to

$$G = \text{Sym}_n.$$

In a few words

1. For three prefix reversals to generate S_n , the two shorter reversal lengths must overlap substantially: A gap that is too large forces intransitivity or imprimitivity.
2. For fixed k , the number of choices of m that can generate S_n is bounded independently of n , and the parity of k determines the exact maximum.
3. There is no “intermediate” primitive subgroup generated by three prefix reversals: once primitivity is established, the generated group must contain the entire alternating group.
4. The generation problem separates into two steps:

prove primitivity

+

check parity.

If at least one generator is odd, then $G = S_n$; if all three are even, then $G = A_n$.

Next: Hamiltonicity

These are Cayley graphs... so they are conjectured to be Hamiltonian.

These are 3-regular Cayley graphs.

Are they Hamiltonian?

Thanks to the NSF for their kind support!

