

Restricted Jacobi Permutations

Permutation Patterns 2026

Kyle Hoffman '27

Davidson College Department of Mathematics and Computer Science

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Collaborators

Based on joint work with:



Alyssa Henke



Derek
Stephens



Yongwei Yuan



Yan Zhuang

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Let S be a finite set of positive integers. A **permutation** $\pi = \pi_1 \pi_2 \dots \pi_n$, where $n = |S|$, is a linear arrangement of the elements of S .

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We let $\mathfrak{S}_n$ be the symmetric group of permutations of length n . We will express permutations in one-line notation;

$$\pi = \pi_1\pi_2 \dots \pi_n.$$

Jacobi permutations are defined recursively [7].

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The empty permutation is Jacobi.

If S is nonempty, let $y = \min S$. Decompose $\pi = \alpha y \beta$. We have that π is Jacobi if α is Jacobi, β is Jacobi, and $|\beta|$ is even.

Let $\mathfrak{J}_n$ denote the set of Jacobi permutations in $\mathfrak{S}_n$; we build them up by length.

$$\mathfrak{J}_0 = \{\varepsilon\}$$

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$$\mathfrak{J}_1 = \{1\}$$

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$$\mathfrak{J}_2 = \{21\}$$

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$$\mathfrak{J}_3 = \{321, 132\}$$

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$$\mathfrak{J}_4 = \{4321, 2431, 4132, 3142, 2143\}$$

Jacobi Permutations

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$$\mathfrak{J}_5 = \left\{ \begin{array}{cccccccc} 54321 & 35421 & 52431 & 42531 & 32541 & 54132 & 53142 & 43152 \\ 52143 & 42153 & 32154 & 15432 & 13542 & 15243 & 14253 & 13254 \end{array} \right\}$$

Let $\mathfrak{S}_n(\sigma)$ be the set of permutations of length n which avoid σ as a pattern.

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Generalizing, let $\mathfrak{S}_n(\Pi)$ be the set of length n permutations avoiding all patterns in the set Π .

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Similarly, $\mathfrak{J}_n(\Pi)$ is the set of length n Jacobi permutations avoiding all patterns in the set Π .

On the other hand, let $j_n(\Pi)$ be the number of length n Jacobi permutations avoiding all patterns in Π .

The **Euler Numbers** are defined as the coefficients of the series expansion $\sec x + \tan x = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}$.

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- By symmetry, down-up alternating permutations
- André permutations [2]
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- and...

- (Surprise!) Jacobi permutations

Jacobi permutations were introduced by Viennot in 1980 in order to give a combinatorial interpretation of the coefficients of the series expansions of Jacobi elliptic functions.

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Viennot proved that for all $n \geq 0$, $|\mathfrak{J}_n| = E_n$ by providing a bijection θ' between Jacobi permutations and alternating permutations [7].

Summary of Pattern Avoidance Results

We achieve a complete enumeration of the Jacobi avoidance classes $\mathfrak{J}_n(\Pi)$ where $\Pi \subseteq \mathfrak{S}_3$

Pattern Avoidance Results: One Pattern

Π	$j_n(\Pi)$	OEIS [5]
$\{213\}$	$\begin{cases} \frac{1}{2n+1} \binom{3n}{n}, & \text{if } n \text{ is even,} \\ \frac{1}{2n+1} \binom{3n+1}{n+1}, & \text{if } n \text{ is odd.} \end{cases}$	A047749
$\{312\}$		
$\{231\}$		
$\{123\}$	$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1} \binom{n-k-1}{k} \binom{n}{2k}$	A101785
$\{132\}$	1	A000012
$\{321\}$	$C_{\lfloor n/2 \rfloor}$ (Catalan numbers)	A000108

Pattern Avoidance Results: Two Patterns

Π	$j_n(\Pi)$	OEIS [5]
$\{123, 213\}$	f_{n-1} (Fibonacci numbers)	A000045
$\{123, 231\}$	$1 + \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$	A033638
$\{123, 312\}$		
$\{213, 231\}$	$2^{\lfloor (n-1)/2 \rfloor}$	A016116
$\{213, 312\}$		
$\{231, 312\}$		

Pattern Avoidance Results: Three or More Patterns

Π	$j_n(\Pi)$	OEIS [5]
$\{123, 213, 231\}$	$\left\lceil \frac{n}{2} \right\rceil$	A065033
$\{123, 213, 312\}$		
$\{123, 231, 312\}$		
$\{213, 231, 312\}$	$\begin{cases} 1, & \text{if } n \geq 2 \text{ is even,} \\ 2, & \text{if } n \geq 3 \text{ is odd.} \end{cases}$	A000034
$\{123, 213, 231, 312\}$		

On a permutation $\pi = \pi_1\pi_2 \dots \pi_n$, we consider several statistics:

- $\text{asc}(\pi)$ = number of ascents, i.e., positions i where $\pi_i < \pi_{i+1}$

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- $\text{lrmin}(\pi)$ = number of left-to-right minima, i.e., positions i where $\pi_i < \pi_j$ for all $j < i$
- $\text{last}(\pi)$ = the last element of π , i.e., π_n

Summary of Results on Distributions of Statistics over $\mathfrak{J}_n(\sigma)$

We also determine the distributions of `asc/des`, `lrmin`, and `last` over $\mathfrak{J}_n(\sigma)$ for each $\sigma \in \mathfrak{S}_3$, as well as over the full set $\mathfrak{J}_n$ of Jacobi permutations [4].

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We also determine the distributions of `asc/des`, `lrmin`, and `last` over $\mathfrak{J}_n(\sigma)$ for each $\sigma \in \mathfrak{S}_3$, as well as over the full set $\mathfrak{J}_n$ of Jacobi permutations [4].

(With the exception of the distribution of `last` over $\mathfrak{J}_n(123)$, which remains open).

A Pattern Avoidance Result

Theorem

For all $n \geq 0$, we have

$$j_{2n}(213) = j_{2n}(312) = \frac{1}{2n+1} \binom{3n}{n}$$

and $j_{2n+1}(213) = j_{2n+1}(312) = \frac{1}{2n+1} \binom{3n+1}{n+1}.$

A Classical Bijection from Permutations to Binary Trees

There is a classical bijection Θ between permutations on S and increasing binary trees on S .

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If π is empty, then $\Theta(\pi)$ is the empty tree.

Otherwise, let $y = \min S$, and write $\pi = \alpha y \beta$. Then, $\Theta(\pi)$ is the increasing binary tree obtained by taking y as the root, taking $\Theta(\alpha)$ as the left subtree of y , and taking $\Theta(\beta)$ as the right subtree.

It is known that $\mathfrak{S}_S$ is in bijection with increasing binary trees on $|S|$ nodes.

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More relevant is the fact that $\mathfrak{S}_S(213)$ and $\mathfrak{S}_S(312)$ are each in bijection with *unlabeled* binary trees on $|S|$ nodes.

We say that an increasing binary tree T on S is a **labeled Jacobi tree** if $T = \Theta(\pi)$ for some Jacobi permutation π of S .

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Let $\mathcal{J}_n$ be the set of unlabeled Jacobi trees on n nodes.

Let $\pi \in \mathfrak{J}_n(213)$ (or $\mathfrak{J}_n(312)$). The previous discussion yields that taking $\Theta(\pi)$ and removing the labels gives a bijection between each of $\mathfrak{J}_n(213)$, $\mathfrak{J}_n(312)$ and $\mathcal{J}_n$.

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Hence, in order to obtain enumerations for $\mathfrak{J}_n(213)$ and $\mathfrak{J}_n(312)$, it suffices to determine $|\mathcal{J}_n|$ for all $n \geq 0$.

Jacobi Trees and Pattern Avoidance

Let $\pi \in \mathfrak{J}_n(213)$ (or $\mathfrak{J}_n(312)$). The previous discussion yields that taking $\Theta(\pi)$ and removing the labels gives a bijection between each of $\mathfrak{J}_n(213)$, $\mathfrak{J}_n(312)$ and $\mathcal{J}_n$.

Hence, in order to obtain enumerations for $\mathfrak{J}_n(213)$ and $\mathfrak{J}_n(312)$, it suffices to determine $|\mathcal{J}_n|$ for all $n \geq 0$.

Theorem

For all $n \geq 0$, we have

$$|\mathcal{J}_{2n}| = \frac{1}{2n+1} \binom{3n}{n} \quad \text{and} \quad |\mathcal{J}_{2n+1}| = \frac{1}{2n+1} \binom{3n+1}{n+1}.$$

Hmm.. That looks familiar

In fact, the formulas which enumerate Jacobi trees on even and odd nodes respectively are precisely those which count ternary trees on n nodes and pairs of ternary trees on n nodes.

Hence, to prove the theorem, we provide a recursive bijection λ between $\mathcal{J}_{2n}$ and ternary trees on n nodes.

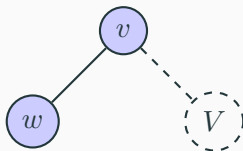
The bijection λ sends the empty tree to the empty tree.

Now, consider a nonempty Jacobi tree T of size $2n$ with root v .



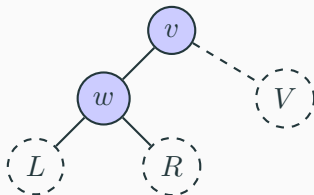
Proof Idea

Since T has even size, v must have a left child w . Let V be the right subtree of v .



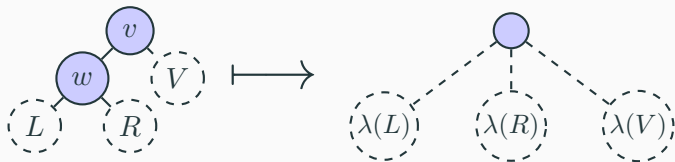
Proof Idea

Let L be the left subtree of w and R be the right subtree of w .
Note that L , R , and V all have even size.



Proof Idea

Define $\lambda(T)$ as a ternary tree with left subtree $\lambda(L)$, middle subtree $\lambda(R)$, and right subtree $\lambda(V)$.



On the other hand, if $T \in \mathcal{J}_{2n+1}$, the bijection is similar.

The Distribution of lrmin over 213/312 Avoiders

We have

Theorem

For all $n \geq k \geq 1$, we have

$$j_{n,k}^{\text{lrmin}}(213) = j_{n,k}^{\text{lrmin}}(312) = \frac{2k}{3n-k} \binom{\frac{3n-k}{2}}{n}$$

if n and k have the same parity, and

$j_{n,k}^{\text{lrmin}}(213) = j_{n,k}^{\text{lrmin}}(312) = 0$ otherwise.

The Distribution of $j_{n,k}^{\text{last}}$ over 213 avoiders

$j_{n,k}^{\text{last}}(213)$					
$n \backslash k$	1	2	3	4	5
1	1				
2	1				
3	1	1			
4	2	1			
5	3	3	1		
6	7	4	1		
7	12	12	5	1	
8	30	18	6	1	
9	55	55	25	7	1

The numbers $j_{n,k}^{\text{last}}(213)$ up to $n = 9$.

The Distribution of lrmin over 213/312 avoiders

$j_{n,k}^{\text{last}}(213)$

$n \backslash k$	1	2	3	4	5
1	1				
2	1				
3	1	1			
4	2	1			
5	3	3	1		
6	7	4	1		
7	12	12	5	1	
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7	12	12	5	1	
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7	12	12	5	1	
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$n \backslash k$									
1	1								
2		1							
3	1		1						
4		2		1					
5	3		3		1				
6		7		4		1			
7	12		12		5		1		
8		30		18		6		1	
9	55		55		25		7		1

The Distribution of lrmin over 213/312 avoiders

$n \backslash k$

1	1							
2		1						
3	1		1					
4		2		1				
5	3		3		1			
6		7		4		1		
7	12		12		5		1	
8		30		18		6		1
9	55		55		25		7	

The Distribution of lrmin over 213/312 avoiders

$n \backslash k$

1	1									
2		1								
3	1		1							
4		2		1						
5	3		3		1					
6		7		4		1				
7	12		12		5		1			
8		30		18		6		1		
9	55		55		25		7		1	

The Distribution of lrmin over 213/312 avoiders

$n \backslash k$

1	1								
2		1							
3	1		1						
4		2		1					
5	3		3		1				
6		7		4		1			
7	12		12		5		1		
8		30		18		6		1	
9	55		55		25		7		1

The Distribution of lrmin over 213/312 avoiders

$n \backslash k$

1	1								
2		1							
3	1		1						
4		2		1					
5	3		3		1				
6		7		4		1			
7	12		12		5		1		
8		30		18		6		1	
9	55		55		25		7		1

The Distribution of lrmin over 213/312 avoiders

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	0	1							
3	0	1	0	1						
4	0	0	2	0	1					
5	0	3	0	3	0	1				
6	0	0	7	0	4	0	1			
7	0	12	0	12	0	5	0	1		
8	0	0	30	0	18	0	6	0	1	
9	0	55	0	55	0	25	0	7	0	1

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0	1									
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3	0	1	0	1						
4	0	0	2	0	1					
5	0	3	0	3	0	1				
6	0	0	7	0	4	0	1			
7	0	12	0	12	0	5	0	1		
8	0	0	30	0	18	0	6	0	1	
9	0	55	0	55	0	25	0	7	0	1

The Distribution of lrmin over 213/312 avoiders

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	0	1							
3	0	1	0	1						
4	0	0	2	0	1					
5	0	3	0	3	0	1				
6	0	0	7	0	4	0	1			
7	0	12	0	12	0	5	0	1		
8	0	0	30	0	18	0	6	0	1	
9	0	55	0	55	0	25	0	7	0	1

The Distribution of $lrmin$ over 213/312 avoiders

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	0	1							
3	0	1	0	1						
4	0	0	2	0	1					
5	0	3	0	3	0	1				
6	0	0	7	0	4	0	1			
7	0	12	0	12	0	5	0	1		
8	0	0	30	0	18	0	6	0	1	
9	0	55	0	55	0	25	0	7	0	1

The numbers $j_{n,k}^{lrmin}(213) = j_{n,k}^{lrmin}(312)$ up to $n = 9$.

The Distribution of j^{last} over 213 avoiders

Proposition

For all $n \geq k \geq 1$, we have

$$j_{n,k}^{\text{last}}(213) = \begin{cases} j_{n,2k}^{\text{lrmin}}(213), & \text{if } n \text{ is even,} \\ j_{n,2k-1}^{\text{lrmin}}(213), & \text{if } n \text{ is odd.} \end{cases}$$

The distribution of j_{last} over 213 avoiders

Then, the following theorem follows by combining the previous proposition with the previous theorem:

Theorem

For all $n \geq k \geq 1$, we have

$$j_{2n,k}^{\text{last}}(213) = \frac{2k}{3n-k} \binom{3n-k}{2n}$$

$$\text{and } j_{2n-1,k}^{\text{last}}(213) = \frac{2k-1}{3n-k-1} \binom{3n-k-1}{2n-1}.$$

Entringer Numbers

The **Entringer Number** $E_{n,k}$ is the number of up-down alternating permutations $\pi \in \mathfrak{U}_n$ with $\pi_1 = k$ [1].

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The **Entringer Number** $E_{n,k}$ is the number of up-down alternating permutations $\pi \in \mathfrak{U}_n$ with $\pi_1 = k$ [1].

Interestingly, $E_{n,k}$ is also the number of the aforementioned André permutations $\pi \in \mathfrak{A}_n$ with $\pi_1 = k$ [2].

Let π be a permutation of length n at least 2, and let π^c denote its **complement**, given by $\pi_i^c = n + 1 - \pi_i$. We define $\text{cpen}(\pi)$ to be the penultimate (second-to-last) letter of π^c .

Example

If $\pi = 263154$, then $\pi^c = 514623$, so $\text{cpen}(\pi) = 2$.

The Distribution of last over $\mathfrak{J}_n$

Theorem

For all $n, k \geq 1$, we have $j_{n,k}^{\text{last}} = E_{n,k}$. Consequently, the distribution of last over $\mathfrak{J}_n$ is equal to the distribution of first over $\mathfrak{U}_n$ or $\mathfrak{A}_n$.

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For the proof, we provide a bijection $\Phi : \mathfrak{J}_n \rightarrow \mathfrak{U}_n$, which sends the last letter of $\pi \in \mathfrak{J}_n$ to the first letter of $\Phi(\pi)$.

The bijection is a modification of a bijection due to Viennot [7].

A stronger conjecture...

We conjecture the following about the joint distribution of cpen and last over $\mathfrak{J}_n$. Let

$$j_{n,i,k}^{(\text{cpen}, \text{last})} := |\{ \pi \in \mathfrak{J}_n : \text{cpen}(\pi) = i \text{ and } \text{last}(\pi) = k \}|.$$

A stronger conjecture...

Conjecture

For all $n \geq 3$ and $i, k \geq 1$, we have $j_{n,i,k}^{(\text{cpen}, \text{last})} = E_{n-2, i+k-2}$.

Another conjecture..

We have

- For all $n \geq 0$, the distribution of `asc` over $\mathfrak{J}_n$ is equal to the distribution of `des` over $\mathfrak{A}_n$.

Another conjecture..

We have

- For all $n \geq 0$, the distribution of asc over $\mathfrak{J}_n$ is equal to the distribution of des over $\mathfrak{A}_n$.

Let $a_{n,i,k}^{(\text{des}, \text{first})}$ be the number of André permutations $\pi \in \mathfrak{A}_n$ of length n with i descents and $\pi_1 = k$.

Another conjecture..

We have

- For all $n \geq 0$, the distribution of asc over $\mathfrak{J}_n$ is equal to the distribution of des over $\mathfrak{A}_n$.

Let $a_{n,i,k}^{(\text{des},\text{first})}$ be the number of André permutations $\pi \in \mathfrak{A}_n$ of length n with i descents and $\pi_1 = k$.

Conjecture

For all $n \geq 1$, $i \geq 0$, and $k \geq 1$, we have $j_{n,i,k}^{(\text{asc},\text{last})} = a_{n,i,k}^{(\text{des},\text{first})}$.

Doubly Jacobi Permutations

A Jacobi permutation π is said to be **doubly Jacobi** if π^{-1} is Jacobi.

Let $\pi \in \mathfrak{S}_n(312)$. It is known that $\pi^{-1} \in \mathfrak{S}_n(231)$.

312, 231 avoiders and inverses

Let $\pi \in \mathfrak{S}_n(312)$. It is known that $\pi^{-1} \in \mathfrak{S}_n(231)$.

Surprisingly, inversion also preserves the Jacobi property.
That is,

Theorem

For all $n \geq 0$, a permutation $\pi \in \mathfrak{S}_n(312)$ is Jacobi if and only if $\pi^{-1} \in \mathfrak{S}_n(231)$ is Jacobi. Consequently, $\pi \mapsto \pi^{-1}$ is a bijection from $\mathfrak{J}_n(312)$ to $\mathfrak{J}_n(231)$.

Theorem

For all $n \geq 0$, the permutation π^{-1} is Jacobi whenever $\pi \in \mathfrak{J}_n(213)$.

Theorem

For all $n \geq 0$, the permutation π^{-1} is Jacobi whenever $\pi \in \mathfrak{J}_n(213)$.

In consequence, all permutations in $\mathfrak{J}_n(312)$, $\mathfrak{J}_n(231)$, and $\mathfrak{J}_n(213)$ are doubly Jacobi.

Doubly Jacobi permutations in general?

It remains open to enumerate doubly Jacobi permutations in $\tilde{\mathfrak{J}}_n$.

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Thank You! Questions?