

Results in chain avoidance

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(Eddie the Golden Eagle)

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Chain avoidance

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Example: The permutation $\pi = 21534$ avoids the chain $(231 : 213, 312 : \emptyset : 4213)$, meaning π avoids 231 , π^2 avoids 213 and 312 , and π^4 avoids 4213 . The $\emptyset$ in the third spot indicates that there are no restrictions on π^3 .

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Example: Permutations avoiding the chain $(\emptyset : 21)$ are involutions.

Avoiding $(\sigma : \sigma)$, aka strong avoidance

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Theorem: The number of permutations in $\mathcal{S}_n$ that strongly avoid 312 is given by coefficients of

$$\frac{1 - x - x^2 + x^3}{1 - 2x - x^2 + 2x^3 - x^4}.$$

To prove it, they used the direct sum structure of 312 avoiding permutations.

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Theorem: The number of permutations in $\mathcal{S}_n$ that strongly avoid 123 is eventually zero.

Avoiding $(132 : 132)$

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$$a(x) = \frac{c(x^3c(x^3))}{2 - (1+x)c(x^3c(x^3))} \left(\frac{1-x}{1-x-x^2c(x^2)} + \frac{2x(1+2x)(c(x^3)-1)}{1-x^2} \right)$$

where $c(x)$ is the generating function for the Catalan numbers.

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The method of proof in this case is to consider the size k of the cycle that contains n :

$$a(x) = 1 + \underbrace{xa(x)}_{k=1} + \underbrace{i_{132}(x)}_{k=2} + \underbrace{\left[2 - \frac{2}{c(x^3c(x^3))} \right] a(x)}_{k=3} + \underbrace{2(c(x^3)-1) \left[\frac{x}{1-x} + \frac{x^2}{1-x^2} \right]}_{k \geq 4}$$

Refined enumeration for (132 : 132)

Question: How many permutations in $\mathcal{A}_n(132)$ have n in a k -cycle?

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	2	5	12	24	50	101	202	398	806	1568	3148	6198	12306
2		1	1	3	4	10	15	35	56	126	210	462	792	1716	3033
3			2	2	4	14	28	56	132	262	524	1098	2202	4316	8858
4				2	-	-	-	4	-	-	-	10	-	-	-
5					4	-	-	-	-	8	-	-	-	-	20
6						2	-	-	-	-	-	4	-	-	-
7							8	-	-	-	-	-	-	36	-
8								6	-	-	-	-	-	-	-
9									8	-	-	-	-	-	-
10										12	-	-	-	-	-
11											28	-	-	-	-
12												6	-	-	-
13													56	-	-
14														40	-
15															36
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Theorem. (A., Graves) Let $A(x)$ be the generating function for permutations avoiding the chain $(231 : 312)$. Then,

$$A(x) = \frac{(1-x)(1-x^2)(1-2x^3)}{1-2x-x^2-x^3+2x^4+3x^5-x^6+x^7-x^8}.$$

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This one is similar to the $(132 : 132)$ case! Looking at the size of the cycle containing n , the only possibilities are sizes 1, 2, 3 and 4. Adding these together gives you the answer above.

Avoiding $(\sigma : \tau)$

Theorem. (A., Graves) Let $a(x)$ be the generating function for the number of permutations avoiding $(231 : 321)$, and let $t(x) = \sum_{n \geq 1} \tau(n)x^n$ where $\tau(n)$ is the number of divisors of n . Then

$$a(x) = \frac{1}{1 - t(x) - q(x)}$$

where

$$q(x) = \frac{2x^9 - 7x^8 + 9x^7 - 7x^6 + 4x^5 - 3x^4 + 3x^3 - x^2}{(1-x)^3(1-x^6)}.$$

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In this case, $t(x) + q(x)$ is the generating function for those permutations avoiding $(231 : 321)$ that start with n .

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From “Powers of permutations that avoid chains of patterns” by A. and Geary (2023):

τ	$c_n(213, 312 : \tau)$	OEIS	τ	$c_n(213, 312 : \tau)$	OEIS
123	$\left\lfloor \frac{n}{2} \right\rfloor$	A004526	231	$\left\lfloor \frac{n(n+1)}{3} \right\rfloor$	A007980
132	$n+1$	A000027	312	$\left\lfloor \frac{n^2}{4} \right\rfloor + 1$	A033638
213	$\left\lfloor \frac{n(n+1)}{3} \right\rfloor$	A007980	321	$\left\lfloor \frac{(n+3)^2}{4} \right\rfloor - 5$	A002620 - 5

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These are easier to deal with because permutations avoiding 213 and 312 (i.e. unimodal permutations) have nice structure, as do their squares.

Avoiding $(312, 321 : \tau)$

From “Pattern avoidance in compositions and powers of permutations” by A. and Bourne (2025):

Theorem: For any τ , you can count permutations avoiding $(312, 321 : \tau)$ using a “Fibonacci-like” recurrence.

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In this case, permutations π always look like:

$$\pi = \bigoplus_{i=1}^m \epsilon_{d_i} \text{ with } \pi^2 = \bigoplus_{i=1}^m \epsilon_{d_i}^2$$

where

- $(d_1, \dots, d_n)$ is a composition of n ,
- $\epsilon_1 = 1$, $\epsilon_2 = 21$, and for $d \geq 3$, $\epsilon_d = 234 \dots d1$, and
- for $d \geq 3$, $\epsilon_d^2 = 34 \dots d12$.

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It is always possible to associate a (nontrivial) τ with a particular set of composition C_τ . It turns out that enumerating permutations avoiding $(312, 321 : \tau)$ is equivalent to enumerating compositions that avoid C_τ .

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Some examples:

τ	C_τ	$a_n(312, 321 : \sigma)$	OEIS
132	$(1,3)$	$F_{n+2} - 1$	A000071
231	(4)	T_{n+2}	A000073
1243	$(1, 1, 3), (2, 3)$	$2F_{n+1} - 2$	A019274
1324	$(1,3,1)$	$F_{n+3} - n - 1$	A000126
2143	$(3,3)$	$1 + \frac{1}{5}(nF_{n+2} + (n-3)F_n)$	A023610 + 1
$234 \dots k1$	$(k+1)$	$F_{n+k-1}^{(k)}$	A000078, etc.

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It's different for $\sigma = 132$! The number avoiding $(132 : \cdots : 132)$ will depend on the number of 132's in the chain.

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Theorem. (Burcroff, Defant) If $A(x)$ is the generating function for permutations that avoid $(231 : \emptyset : 21)$, then

$$A(x) = \frac{1 - 2x^3}{1 - x - 3x^3 - x^5}.$$

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Theorem. (A., Laudone) If $B(x)$ is the generating function for permutations that avoid $(132 : \emptyset : 21)$, then

$$B(x) = \frac{c(x^3)}{\sqrt{c(x^3)(4 - 3c(x^3))} - xc(x^3)}.$$

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Open Questions!

- How many permutations avoid $(321 : 321)$?
- How many permutations avoid $(\sigma : \tau)$ with $\sigma \neq \tau$ and $\sigma \notin \{312, 231\}$.
- What about π and π^2 avoiding (other) sets of permutations?
- How many permutations have the property that $\text{des}(\pi) = k$ and $\text{des}(\pi^2) = j$? How about permutations whose square has at most one descent?

References

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Thanks!

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