Permutations containing a fixed number of occurrences of a pattern

Michael Waite
University of Florida

Permutation Patterns 2026
July 23, 2026

Basics

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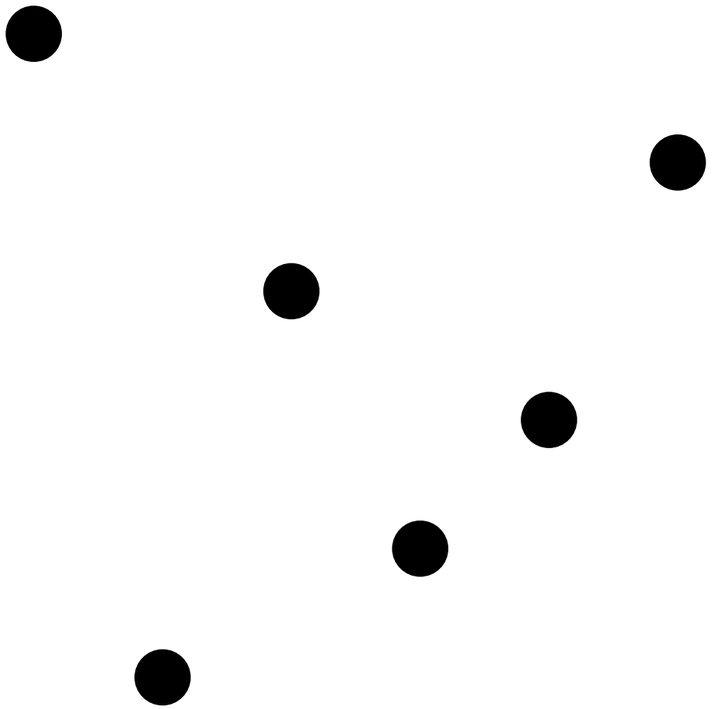

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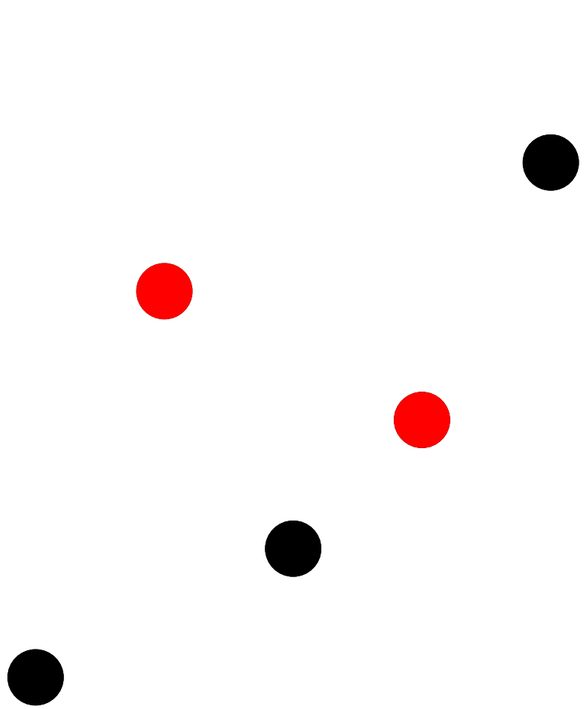

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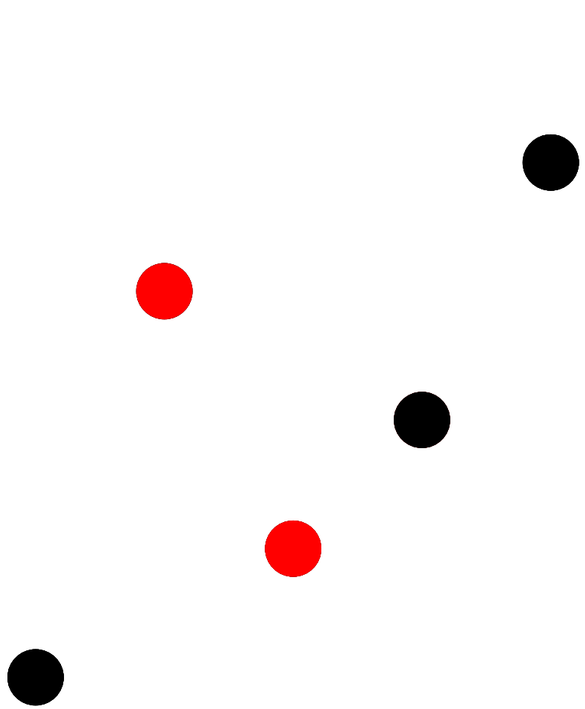

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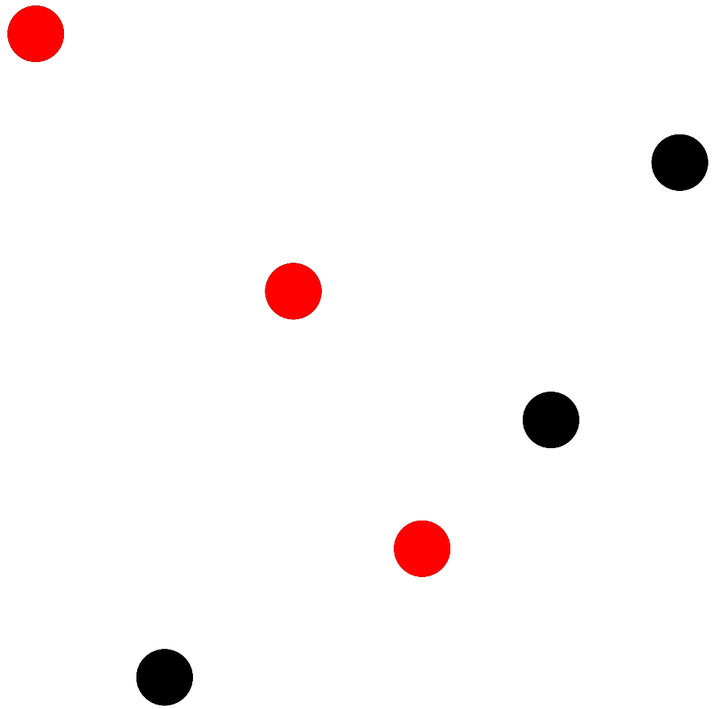

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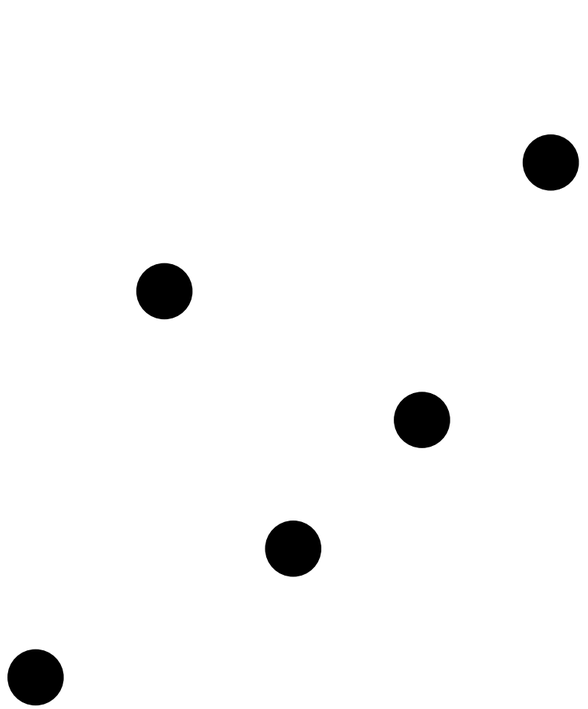

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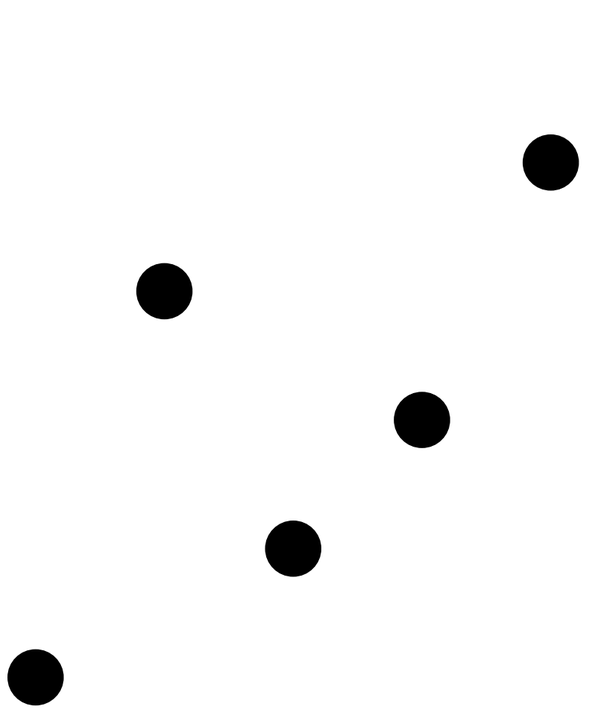

How many copies of 321?

2, so $p \in S_{6,2}(321)$.

How many copies of 231?

0, so $p \in S_6(231)$.

Known Results

	$\lvert S_{n,r}(321)\rvert$	$\lvert S_{n,r}(231)\rvert$
$r=1$	$\frac{3}{n}\binom{2n}{n-3}$ (Noonan 1996)	$\binom{2n-3}{n-3}$ (Bóna 1997)

Known Results

	$\lvert S_{n,r}(321)\rvert$	$\lvert S_{n,r}(231)\rvert$
$r=1$	$\frac{3}{n}\binom{2n}{n-3}$ (Noonan 1996)	$\binom{2n-3}{n-3}$ (Bóna 1997)
$r=2$	$\frac{59n^2+117n+100}{2n(n-1)(n+5)}\binom{2n}{n-4}$ (Fulmek 2001)	$\frac{n^3+17n^2-80n+80}{2n(n-1)}\binom{2n-6}{n-4}$ (Mansour, Vainshtein 2002)

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$r=3$	Enumeration known (Callan 2002)	Enumeration known (Mansour, Vainshtein 2002)
$r=4$	Enumeration known (Callan 2002)	Enumeration known (Mansour, Vainshtein 2002)
$r=5$		Enumeration known (Mansour, Vainshtein 2002)
$r=6$		Enumeration known (Mansour, Vainshtein 2002)

Known Results

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Theorem (Bóna 1997)
Let $r \geq 1$. Let $S_r(z) = \sum_{n \geq 0} |S_{n,r}(231)|$. Then

$$S_r(z) \in \mathbb{C}(z, \sqrt{1-4z}).$$

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For everything else, nothing so general is known about the generating functions.

Negative Results

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Theorem
Let $f(z) = \sum_{n \geq 0} f_n z^n$ be a generating function. If α **is not a non-negative integer (is a negative integer)** and for all positive integers n the chain of inequalities

$$kn^{\alpha}\gamma^n \leq f_n \leq Kn^{\alpha}\gamma^n$$

holds, then $f(z)$ is **not a rational (is not an algebraic)** power series.

Negative Results

This means that if we can find bounds

$$k_r n^\alpha |S_n(q)| \leq |S_{n,r}(q)| \leq K_r n^\alpha |S_n(q)|$$

and if we know the asymptotics of $|S_n(q)|$, then we can make conclusions about the generating function for $|S_{n,r}(q)|$.

The Trivial Bound

Proposition
Let $r > 0$ and let q be any permutation. Then there are constants k_r and K_r so that

$$k_r|S_n(q)| \leq |S_{n,r}(q)| \leq K_r n^{2r}|S_n(q)|.$$

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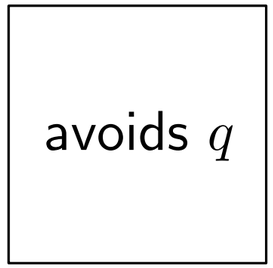

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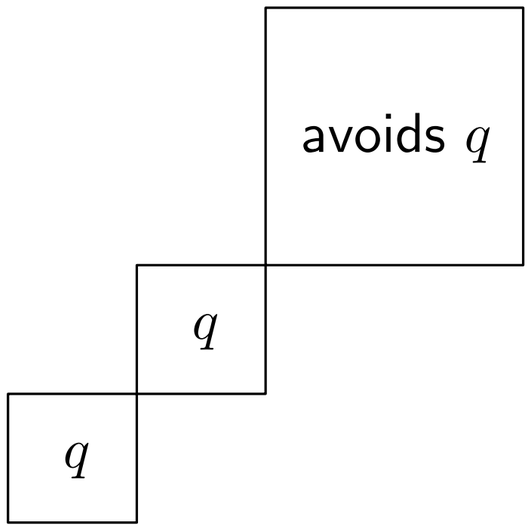

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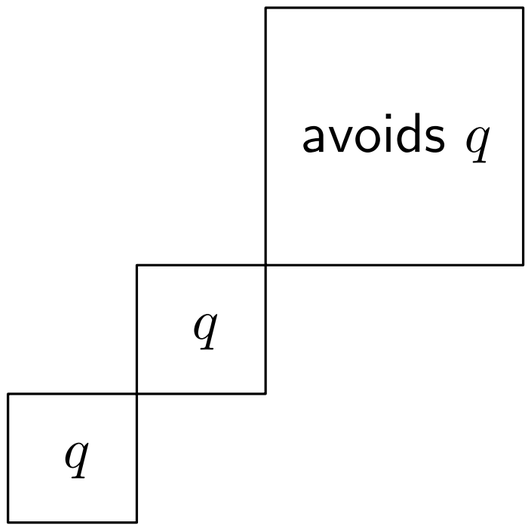

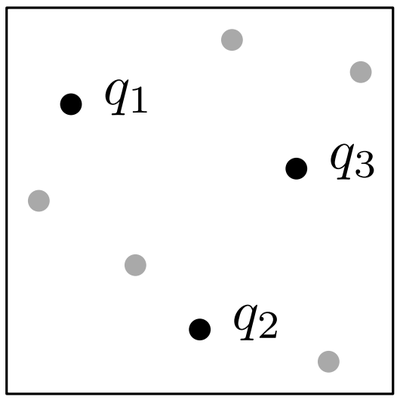

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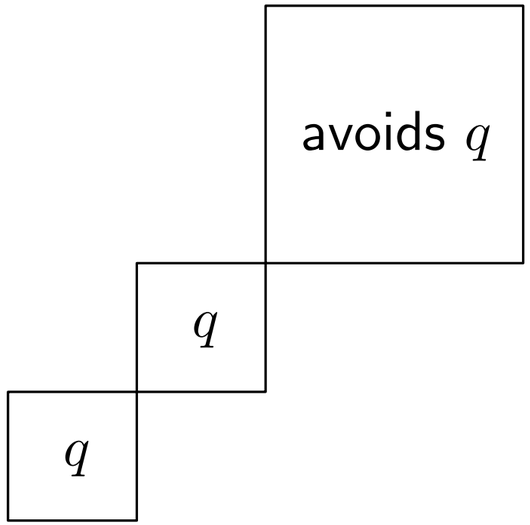

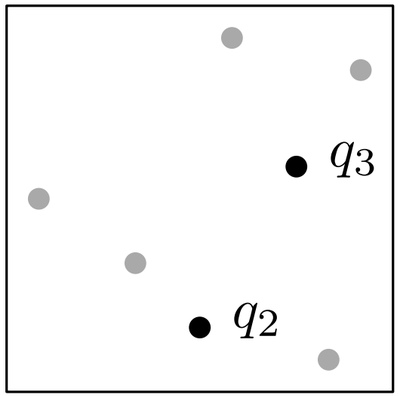

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An Upper Bound for 321

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Let $r > 0$. Then there is a constant K_r so that

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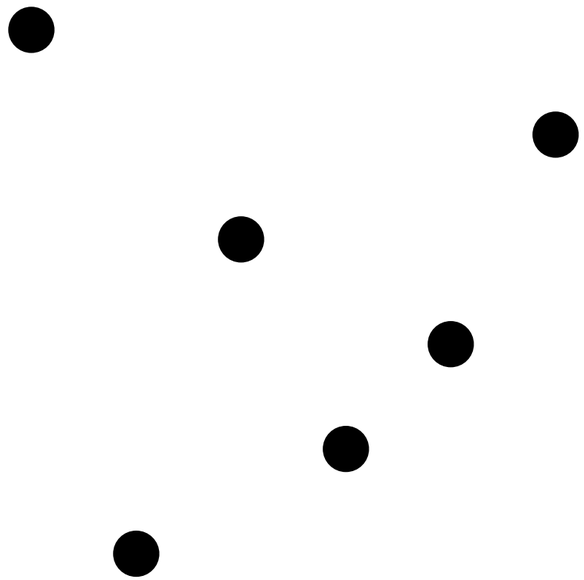

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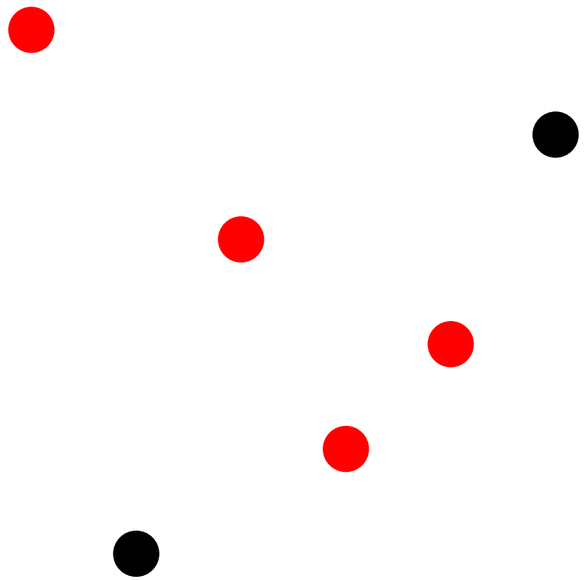

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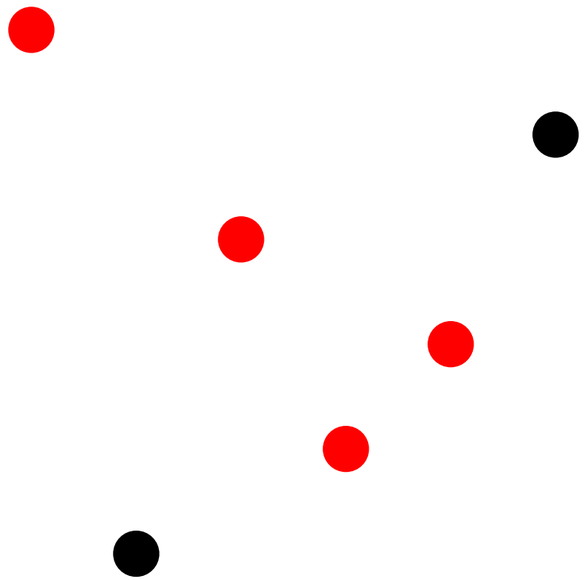

So $p \in S^{4312}_{6,2}(321)$.

An Upper Bound for 321

Now for each q, we will construct an injection

$$\phi^q : S^q_{n,r}(321) \hookrightarrow \bigcup_{k=1}^{n} S_{k-1+i,s}(321) \times S_{n-k+j,t}(321)$$

for some constants i, j and for some $s, t < r$.

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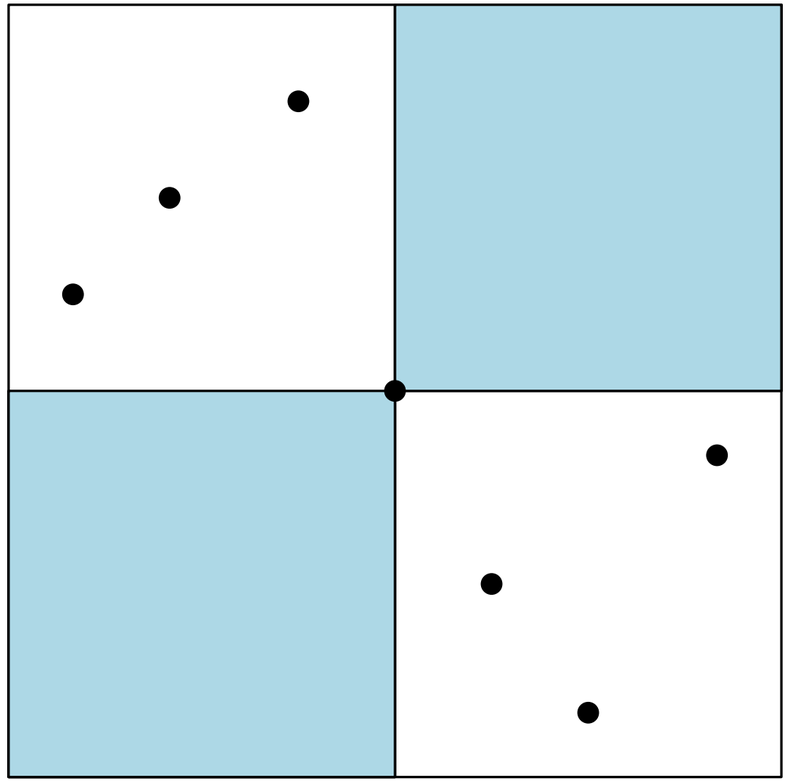
Since there are only finitely many possible q, we can sum over all possible q and get

$$|S_{n,r}(321)| \leq C_r|S_n(321)|.$$

The Map



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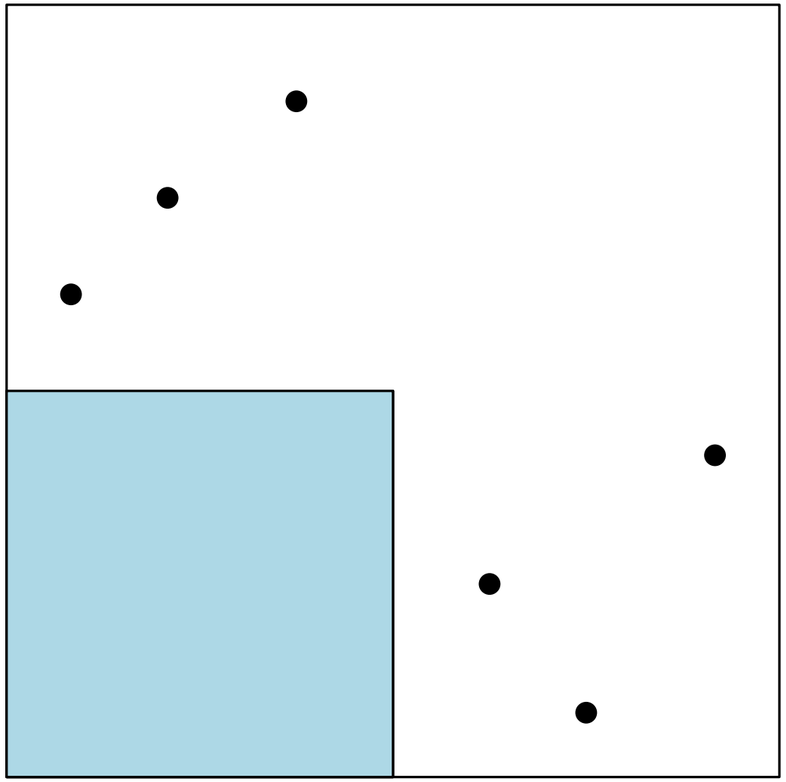

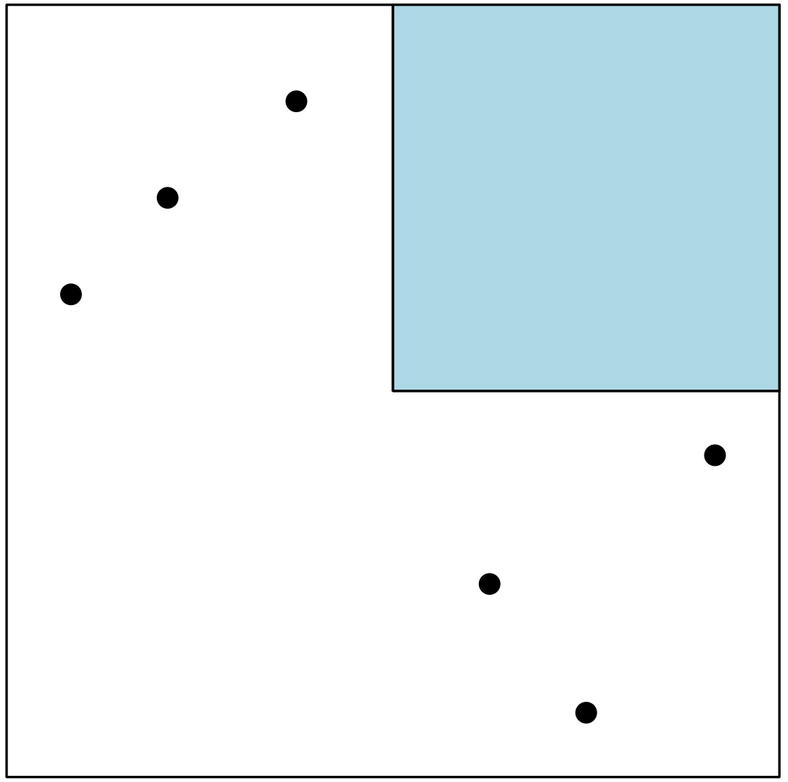

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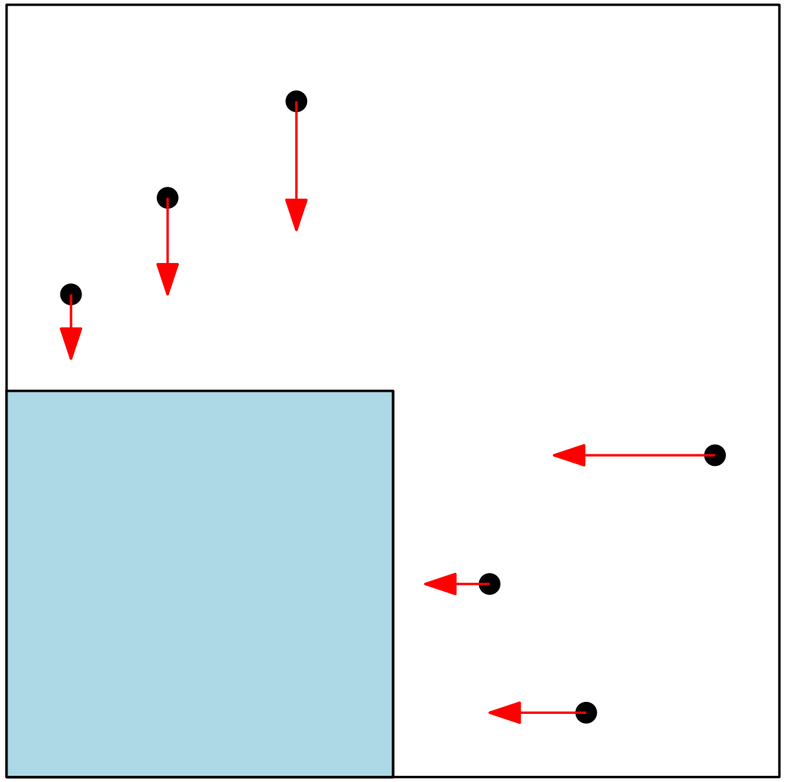

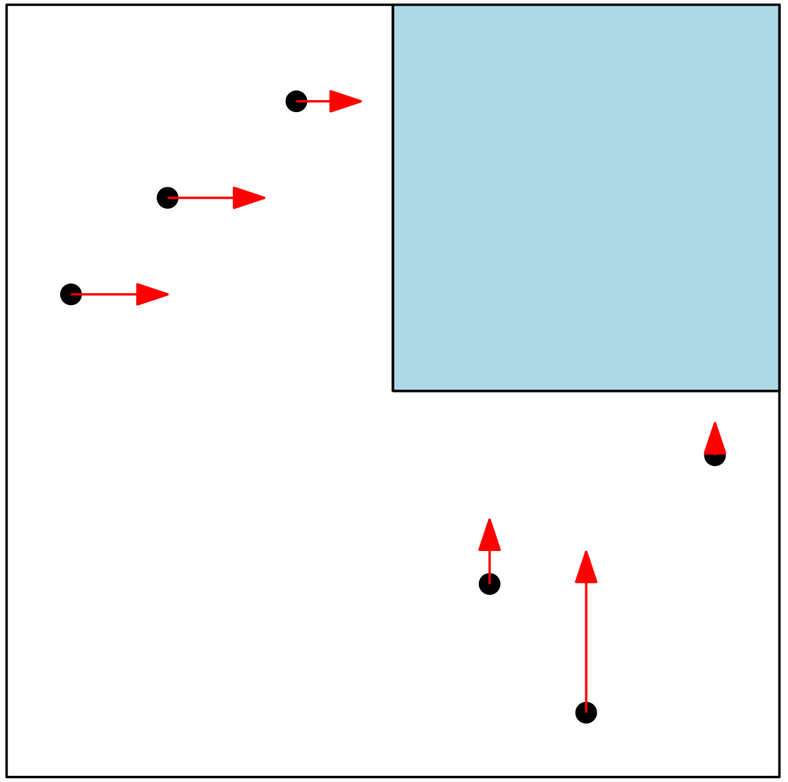

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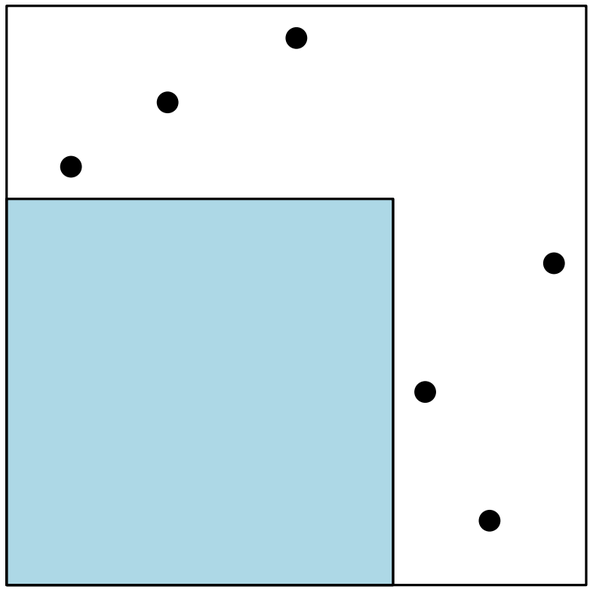

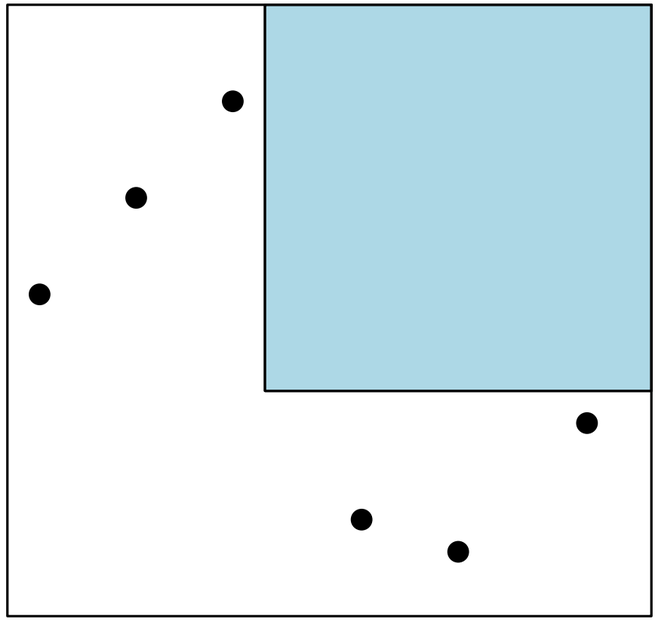

A Generalization

Theorem (W 2026+)
Let q be a permutation, and let $r \geq 1$. Then there are constants k_r, K_r so that for all n sufficiently large we have

$$k_r |S_n(321 \ominus q)| \leq |S_{n,r}(321 \ominus q)| \leq K_r |S_n(321 \ominus q)|.$$

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Corollary (W 2026+)
Suppose $k \geq 3$. There is $c_{k,r}$ and $C_{k,r}$ such that

$$c_{k,r} \frac{(k-1)^{2n}}{n^{((k^2-2k)/2)}} \leq |S_{n,r}(k(k-1)...1)| \leq C_{k,r} \frac{(k-1)^{2n}}{n^{((k^2-2k)/2)}}.$$

Let $S_r(z) = \sum_{n \geq 0} |S_{n,r}(q)| z^n$. Then:

- If k is odd, then $S_r(z)$ is not rational.
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Bóna and Burstein (2021) proved this result for $r = 1$.

Further Results

Theorem (W 2026+)
Let q be a separable indecomposable permutation of length at least 2 which is of the form $q_1 \ominus ... \ominus q_k$ for some $k \geq 2$ where all of the q_i are skew-indecomposable and $q_2, ..., q_{k-1}$ are not 1. Then there are constants k and K such that for all n sufficiently large we have

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Corollary (W 2026+)
Suppose $r \geq 1$. Then there is c_k and C_k such that

$$c_k \frac{(k-1)^{2n}}{n^{((k^2-2k)/2)-r}} \leq |S_{n,r}(23...k1)| \leq C_k \frac{(k-1)^{2n}}{n^{((k^2-2k)/2)-r}}.$$

Let $S_r(z) = \sum_{n\geq 0} |S_{n,r}(23...k1)|z^n$. Then:

- If k is odd, then $S_r(z)$ is not rational.
- If k is even and $r < \frac{k^2-2k}{2}$, then $S_r(z)$ is not algebraic.

Further Results

Corollary (W 2026+)
Suppose $r \geq 1$. Then there is c and C such that

$$c\frac{9^n}{n^{4-r}} \leq |S_{n,r}(3412)| \leq C\frac{9^n}{n^{4-r}}.$$

Let $S(z) = \sum_{n\geq 0} |S_{n,r}(3412)|z^n$. If $r < 4$, then $S(z)$ is not algebraic.

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Suppose $r \geq 1$. Then there is c and C such that

$$c\frac{8^n}{n^{5/2-r}} \leq |S_{n,r}(4213)| \leq C\frac{8^n}{n^{5/2-r}}.$$

Let $S(z) = \sum_{n\geq 0} |S_{n,r}(4213)|z^n$. Then $S(z)$ is not rational.

Further Directions

Conjecture (Conway and Guttmann 2023)
Let $q \in \{4321, 4312\}$ and $r \geq 1$. Then there is a constant K so that

$$|S_{n,r}(q)| \sim K|S_n(q)|.$$

Let $q \in \{4123, 3412, 4231, 4213, 3142\}$ and $r \geq 1$. Then there is a constant K so that

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Let $q \in \{4321, 4312\}$ and $r \geq 1$. Then there is a constant K so that

$$|S_{n,r}(q)| \sim K|S_n(q)|.$$

Let $q \in \{4123, 3412, 4231, 4213, 3142\}$ and $r \geq 1$. Then there is a constant K so that

$$|S_{n,r}(q)| \sim Kn^r|S_n(q)|.$$

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What can be said about $|S_{n,r}(4312)|$?

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In every case we have observed, either $|S_{n,r}(q)| = \Theta(|S_n(q)|)$ or $|S_{n,r}(q)| = \Theta(n^r|S_n(q)|)$. But it seems unlikely that this will always be true.

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Question
Is there a pattern q so that $|S_{n,r}(q)| = \Theta(n^{\alpha r}|S_n(q)|)$ for any $\alpha \notin \{0, 1\}$?

Thank you for listening!

Questions?