Comparability in Bruhat Orders

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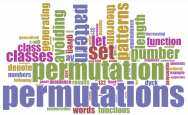
Permutation Patterns, Cal State LA · July 23, 2026

LACIM

Laboratoire d'algèbre, de combinatoire et d'informatique mathématique



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- A permutation is determined by its inversions. Inclusion of inversions defines a poset on permutations.
- The one-skeleton of the permutohedron can be viewed as a Hasse diagram for a poset on permutations. [Ziegler '95].
- Say $\sigma \preceq_W \tau$ if σ appears in the process of applying a bubble sort to τ. [Knuth '98]

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- Schubert cells Ω_σ° in GL_n/B are indexed by permutations. Say $\sigma \preceq_B \tau$ if the closure of Ω_τ° contains Ω_σ°. [Ehresmann '34; Chevalley '94]
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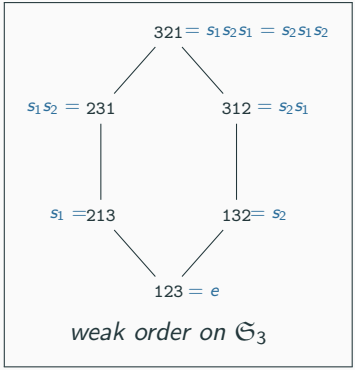
Weak Bruhat order

Write $\tau = s_{i_1} \cdots s_{i_k}$ as a *reduced* product of simple transpositions.

Definition (weak order)

$\sigma \preceq_W \tau$ iff σ is a **prefix** of some reduced word for τ: $\sigma = s_{i_1} \cdots s_{i_m}$, $m \leq k$.



weak order on $\mathfrak{S}_3$

Weak Bruhat order

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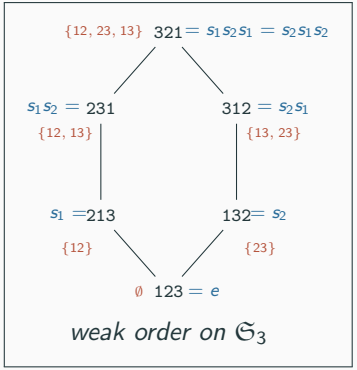
- Let $\mathrm{Inv}(\sigma)$ be the set of pairs (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$. Then, [Björner–Wachs '91]: $\sigma \preceq_W \tau \iff \mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\tau)$.
- Going *up* an edge = swapping an adjacent pair into an *inversion*.
  - Cover relations: an occurrence of the mesh pattern becomes an occurrence of the mesh pattern .



$\{ij\}$ next to σ is $\mathrm{Inv}(\sigma)$

A basic question about any finite poset

Given a poset $(P, \preceq)$, pick two elements independently and uniformly at random.

How likely are they comparable? $\mathbb{P}(X \preceq Y) = \dfrac{\#\{\text{intervals}\}}{|P|^2}$.

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For many classical posets this is exactly solvable:

chain C_N $\binom{N+1}{2}$ $(\mathbb{P} \to \frac{1}{2})$

Boolean B_N 3^N $(\mathbb{P} = (3/4)^N)$

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Naive bounds

$$n! \;\le\; |\mathcal{W}_n| \;\le\; (n!)^2, \qquad \text{i.e.}$$
$$(1+o(1))\, n \log n \;\le\; \log |\mathcal{W}_n| \;\le\; (2+o(1))\, n \log n.$$

The Big Question

$$(1+o(1))\, n \log n \;\le\; \log |\mathcal{W}_n| \;\le\; (2+o(1))\, n \log n$$

Hammett–Pittel (2008) [HP08] studied this and proved

$$(n!) \prod_{i=1}^{n} H(i) \;\le\; |\mathcal{W}_n| \;\le\; (n!)^2 (0.362)^n, \qquad H(i) = \textstyle\sum_{j \le i} \frac{1}{j}.$$

Taking logarithms,

$$(1+o(1))\, n \log n + \log\log n \le \log |\mathcal{W}_n| \;\le\; (2+o(1))\, n \log n + \log(0.362)\, n.$$

Better than the naive bounds, but does not capture leading order behaviour.

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A natural problem is to find the **coefficient of** $n\log n$: a constant $c_W \in [1,2]$ with

$$\log|\mathcal{W}_n| = (c_W+o(1))\, n\log n.$$

Main result

Theorem (B.–Cornejo–Hodges–Horn–Lesnevich–McAllister)

As $n \to \infty$,

$$|\mathcal{W}_n| = (n!)^2 \exp\Big(-\big(\tfrac{1}{2} + o(1)\big)\, n \log n\Big), \qquad \mathbb{P}(\sigma_1 \preceq_W \sigma_2) = \exp\Big(-\big(\tfrac{1}{2} + o(1)\big)\, n \log n\Big).$$

Equivalently, c_W is determined exactly:

$$\log |\mathcal{W}_n| = \big(\tfrac{3}{2} + o(1)\big)\, n \log n \approx (n!)^{3/2}$$

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On the logarithmic scale, this is **exactly halfway** between the naive bounds $n!$ and $(n!)^2$.

Proof sketch, step 1: reduce to typical permutations

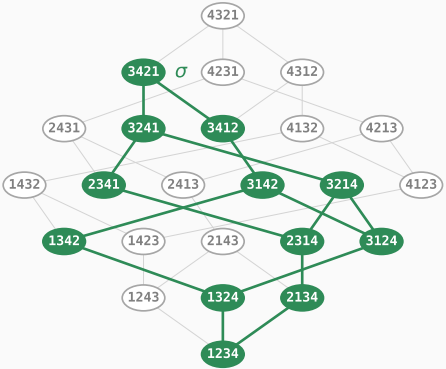
Count intervals by their top element:

$$|\mathcal{W}_n| \;=\; \sum_{\sigma \in \mathfrak{S}_n} e(\sigma), \qquad e(\sigma) := \#\{\pi : \pi \preceq_W \sigma\}$$

is the size of the **down-set** of σ. We are left to compute:

how large is $e(\sigma)$ for a typical σ?

down-set of $\sigma = 3421$ in $\mathfrak{S}_4$: $e(\sigma) = 12$



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We will show: Typically,

$$e(\sigma) = (n!)^{1/2+o(1)} \approx \sqrt{n!},$$

hence $|\mathcal{W}_n| \approx n! \cdot \sqrt{n!} = (n!)^{3/2}$.

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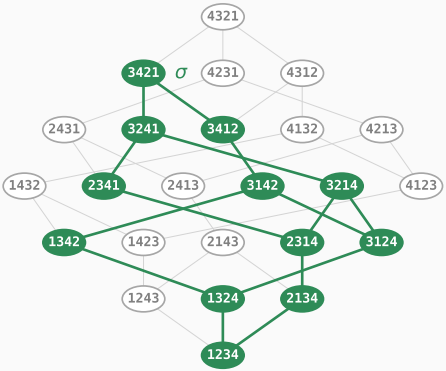
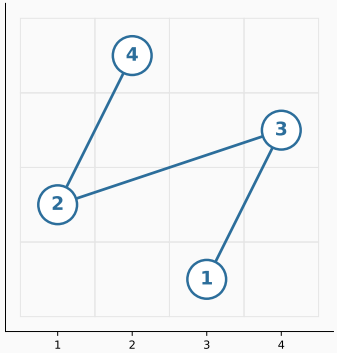


Step 2: the poset $P(\sigma)$

From σ, build the poset $P(\sigma)$ on positions $\{1, \ldots, n\}$:

$$i \prec j \quad \Longleftrightarrow \quad i < j \ \text{ and } \ \sigma(i) < \sigma(j).$$

Its *chains* are exactly the increasing subsequences of σ. Comparable elements are non-inversions.



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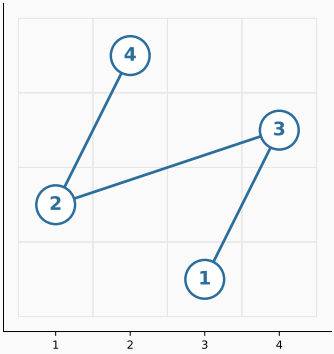
Key fact

Since weak order is containment of inversions, $e(\sigma)$ equals the number of *linear extensions* of $P(\sigma)$.



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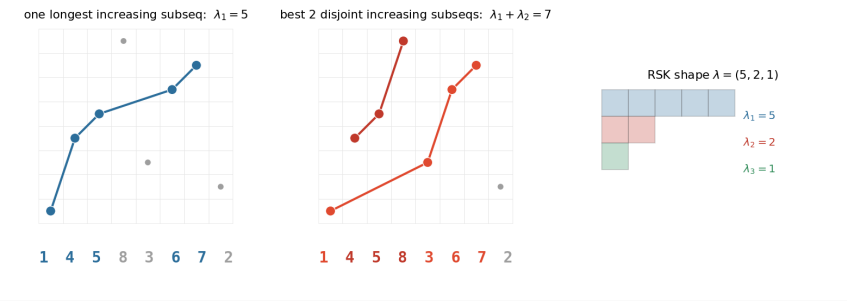
Step 3: RSK and Greene's theorem

Greene's theorem [Greene '74]. For the RSK shape $\lambda(\sigma)$, the sum $\lambda_1 + \cdots + \lambda_k$ is the largest total size of k disjoint chains of $P(\sigma)$ (increasing subsequences).



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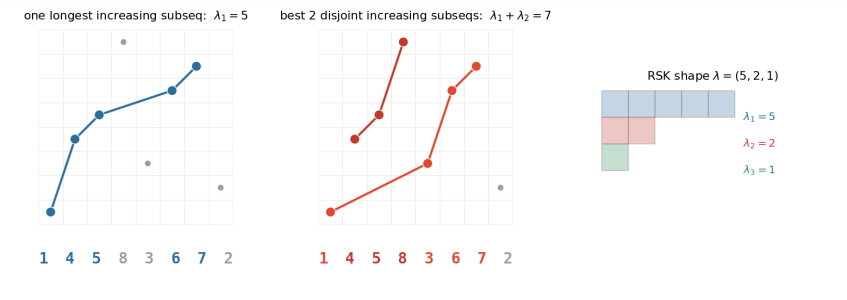
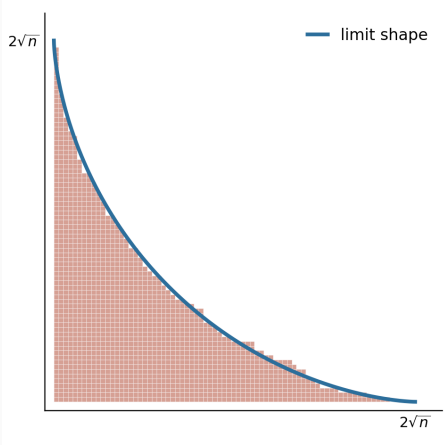


Heuristic. If $P(\sigma)$ were a disjoint union of chains of lengths $\lambda_1, \ldots, \lambda_r$, it would have $\frac{n!}{\lambda_1! \cdots \lambda_r!}$ linear extensions; so $\log e(\sigma) \approx n \log n - \sum_i \lambda_i \log \lambda_i$. An appeal to the Bochkov–Petrov inequalities [Bochkov–Petrov '21] makes this precise.

Step 4: typical shape in Plancherel measure

For uniform σ, the shape $\lambda(\sigma)$ follows the Plancherel measure and concentrates on a fixed limit shape [Logan–Shepp '77; Vershik–Kerov '77]. Typically every part is $\lambda_i = \Theta(\sqrt{n})$ — no long rows or columns [Baik–Deift–Johansson '99].



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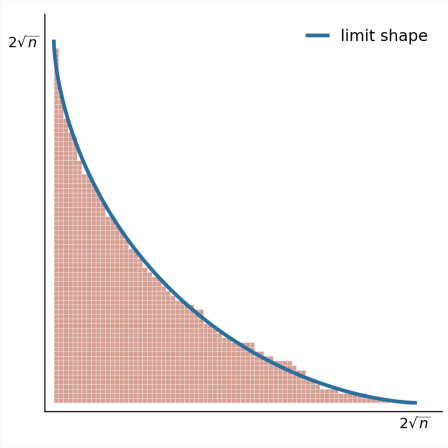

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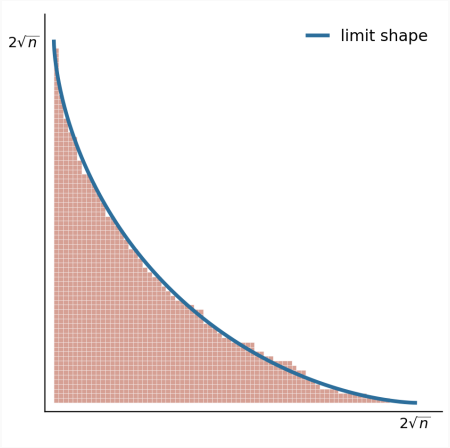
Computing $\mathcal{W}_n$

$$\log e(\sigma) \approx n \log n - \textstyle\sum_i \lambda_i \log \lambda_i \approx \frac{1}{2} n \log n \;\Rightarrow\; e(\sigma) \approx (n!)^{1/2},$$
$$|\mathcal{W}_n| \approx n!\, e(\sigma) \approx (n!)^{3/2}.$$



Strong Bruhat order

Again, write $\tau = s_{i_1} \cdots s_{i_k}$ as a *reduced* product of simple transpositions.

Definition (weak order)

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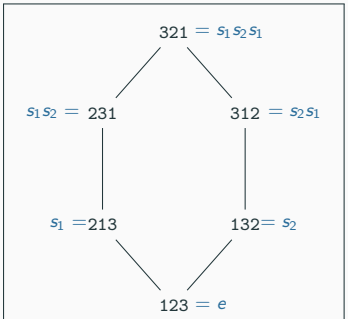

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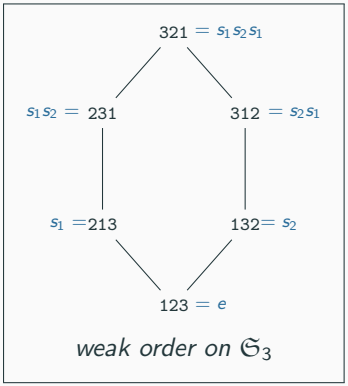

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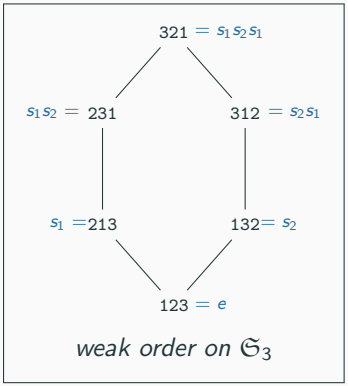

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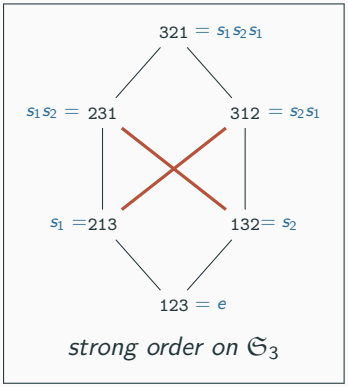

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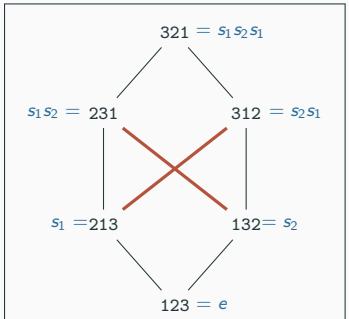
$$\sigma = s_{i_{j_1}} \cdots s_{i_{j_m}}, \quad j_1 < \cdots < j_m$$



strong order on $\mathfrak{S}_3$

- Weak refines strong: $\sigma \preceq_W \tau \implies \sigma \preceq_B \tau$.
- If $\sigma \preceq_B \tau$, then in fact *any* reduced expression for τ contains a subword for σ.
- Cover relations: an occurrence of the mesh pattern becomes an occurrence of the mesh pattern .

Strong order and convergent research

Hammett–Pittel gave an *exponentially* decaying lower bound $(0.708)^n = \exp(\log(0.708)n)$; we improve it to a **subexponential** one:

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$$\mathbb{P}(\sigma_1 \preceq_B \sigma_2) \;\geq\; \exp\big(-(6+o(1))\sqrt{n}\,(\log n)^{3/2}\big) \;=\; \exp\big(-n^{1/2+o(1)}\big).$$

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Theorem (Christo–Michelen)

$$\mathbb{P}(\sigma_1 \preceq_B \sigma_2) = \exp\big(-\Theta(\log^2 n)\big).$$

Also probabilistic, using a Gaussian process to model a *two-dimensional* random walk.

Open Problems

- Can we get asymptotics for other types?
- Can the asymptotics be refined by length of interval?
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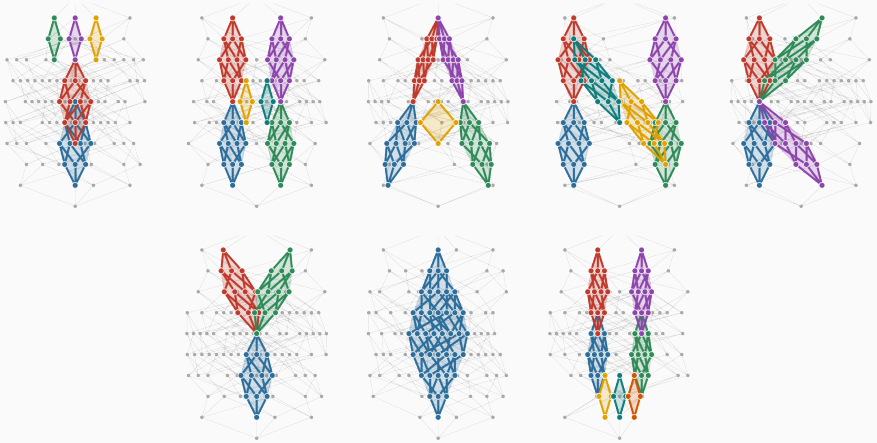

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A cautionary tale...

Thank you!

each letter: a copy of weak order on $\mathfrak{S}_5$, with several *intervals* (distinct colors) shaded in.

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