

Inverse limits of posets from pattern avoidance, set partitions

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Abstract: It is known when we call a poset P , a $\mathcal{P}$ -chain permutational poset, given a subset of permutations $\mathcal{P}$ of the symmetric group S_n . Here we use that idea to study subsets of words of length n , that are not necessarily permutations, for example: r.g.f.'s (restricted growth functions) over $[n] = \{1, 2, \dots, n\}$. Varying n only, and also varying n and k (the number of blocks of the set partitions of $[n]$), we can show that those posets form a projective system of trees and lattices. These lattice structures can be extended over standard type B set partitions over $\langle n \rangle = \{-n, -(n-1), \dots, -2, -1, 0, 1, 2, \dots, n\}$ and in other algebraic and combinatorial scenario. We study properties of the tree and lattice from these projective systems and their inverse limits.

Outline:

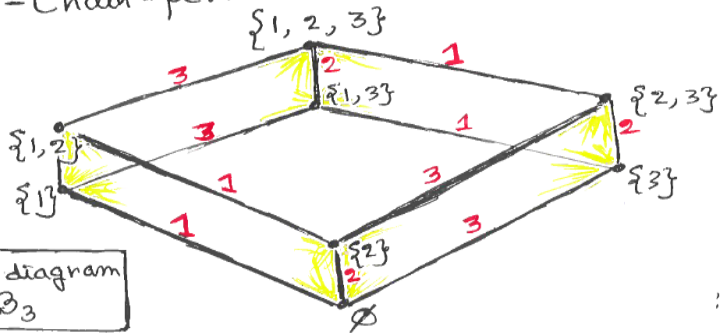
1. Chain-Permutational Posets (Example-Boolean Poset)
2. Chain-Word Posets (Example-Boolean Poset)
3. Restricted Growth Functions (R.G.F.)
4. Pattern Avoidance
5. Chain-Word posets by Pattern Avoidance Classes via R.G.F.
6. Projective Fraisse Family of trees (and lattices) with monotone epomorphisms by chain word posets via R.G.F.
7. Projective Fraisse Family as above for Type B set partitions via signed restricted growth functions (S.R.G.F.)
8. Projective Fraisse Family as above for non crossing partitions

Definition

[2] A poset P will be called $\mathcal{P}$ - chain - permutational, where $\mathcal{P} \subseteq S_n$, if it is possible to label the **covering relations of P** (i.e. the edges of the Hasse diagram of P) with numbers from $1, 2, \dots, n$ in a way that along **different maximal chains of P** (whose length is necessarily n) the labels form different permutations from $\mathcal{P}$, and every $\sigma \in \mathcal{P}$ arises in this manner.

For example, it is well known that the **Boolean lattice B_n is an S_n chain permutational poset**, where $\mathcal{P}$, is the set of all permutations in S_n in one line notation.

Example :- Boolean lattice B_n is
 S_n -Chain-permutational.



Definition

[1] A poset P will be called $\mathcal{P}$ - chain - word, where $\mathcal{P}$ is a subset of the set of all words of length n , if it is possible to label the **covering relations of P** (i.e. the edges of the Hasse diagram of P) with numbers from $1, 2, \dots, n$ in a way that along **different maximal chains of P** (whose length is necessarily n) the labels form different words from $\mathcal{P}$, and every word w of $\mathcal{P}$ arises in this manner.

For example, it is well known that the Boolean lattice B_n is a chain word poset, where $\mathcal{P}$, the set of words is all the words (here permutations), in one line notation in S_n .

Definition

[3] An **R.G.F.** is a sequence $w = a_1 a_2 \cdots a_n$ of positive integers subject to

1. $a_1 = 1$
2. For $i \geq 2$, $a_i \leq 1 + \max\{a_1, a_2, \dots, a_{i-1}\}$

We assume all partitions $\sigma = B_1/B_2/\cdots/B_k \in \Pi_n$ is in standard form, that is $\min(B_1) < \min(B_2) < \cdots \min(B_k)$

There is a **bijection** in between Π_n and R_n , where

Π_n is the set of all set partitions of $[n]$

R_n is the set of all restricted growth functions of length n .

The bijection is as follows:

Let $\sigma = B_1/B_2/\cdots/B_k \in \Pi_n$

then $w(\sigma) = a_1 a_2 \cdots a_n$, where $a_i = j$ iff $i \in B_j$ Example: If

$\sigma = 13/25/467 \in \Pi_7$, then

$w(\sigma) = 1213233 \in R_7$

We standardize set partition with integral elements by replacing the smallest element by 1, next smallest by 2 and so forth. For example, standardization of $25/4$ is $13/2$

Definition

Given two set partitions σ and π , we say that σ contains π as a pattern, if $\exists$ a subpartition of σ which standardizes to π . Otherwise, we say that σ avoids π .

For example: Standardization of $12/345/67 \in \Pi_7$ avoids $13/2$

Thus the above natural bijection is preserved in between

1. $\Pi_n(\pi) = \{\sigma \in \Pi_n : \sigma \text{ avoids } \pi\}$ and
 $R_n(\pi) = \{w(\sigma) : \sigma \in \Pi_n(\pi)\}$
2. $\Pi_{n,k}(\pi)$ and $R_{n,k}(\pi)$, where $\Pi_{n,k}$ is the set of all partitions of $[n]$ in k -many blocks and $R_{n,k}$ is the set of all r.g.f.s of length n with largest integer element k .

As in [1] for any positive integer n , there is a natural way to create a rooted n -ary tree structure (viewed as a poset) labeling the top edge from the root always as 1 (the first digit of an r.g.f.'s being 1). Along different maximal chains (whose length is necessarily n) of the tree, the labels form different words from R_n .

On the other hand, every word w of R_n arises in this manner, displaying any R_n as a $\mathcal{P}$ -chain word poset, where $\mathcal{P} = R_n$. To avoid confusion in notation, we denote the corresponding poset P also by R_n . We see a natural embedding of R_n inside any R_m , where $m > n$ as mentioned in section 6 in [2].

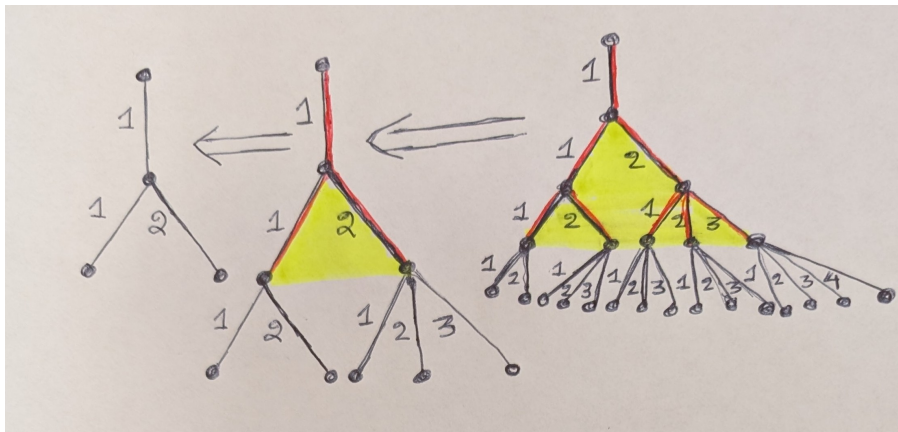


Figure: $R_2 \subseteq R_3 \subseteq R_4$ (as highlighted in the top right)

Definition

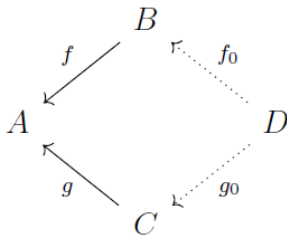
Definition 1.17 [2](Ref P-2) Let F be a nonempty family of finite graphs or a family of **finite rooted graphs** with a fixed family of **epimorphisms** among the structures in F . We say that F is a **Projective Fraisse Family** if

1. F is **countable** up to isomorphism, that is, any sub-collection of pairwise non isomorphic structures of F is countable.
2. Epimorphisms in F are **closed under composition** and each identity map is in F .
3. For $B, C \in F$, $\exists D \in F$ and epimorphisms f and g in F such that $f : D \mapsto B$ and $g : D \mapsto C$.

the definition continues:

1. For every two epimorphisms $f : B \twoheadrightarrow A$ and $g : C \twoheadrightarrow A$ in F there exist epimorphisms $f_0 : D \twoheadrightarrow B$ and $g_0 : D \twoheadrightarrow C$ in F such that

$$f \circ f_0 = g \circ g_0$$



Definition

Definition 3.2, [2], (Ref P-2) Given two topological graphs G and H an epimorphism $f: G \mapsto H$ is called **monotone** if the preimage of a closed connected subset of H is a connected subset of G .

Lemma

Lemma 3.4, [2], (Ref P-2) Let G and H be topological graphs and let $f: G \mapsto H$ be an epimorphism. If the preimage of every vertex in H is connected, then f is monotone.

Definition

Definition 4.1, [2], (Ref P-2) Given two topological graphs G and H , an epimorphism $f: G \mapsto H$ is called **confluent** if for every closed connected subset Q of $V(H)$ and every component C of $f^{-1}(Q)$ we have $f(C) = Q$.

In [1], we find a **natural projection of the Hasse diagram of R_m onto R_n** , as an epimorphism of graphs and posets, ($m > n$), creating a **Projective Fraisse Family of rooted trees** (and a poset, elements of the posets are the vertices of the trees) with **monotone epimorphisms**. (Since, R_{n+1} contains a copy of R_n in it's top and the additional edges in the bottom level of R_{n+1} can be mapped onto the corresponding loops based at the bottom vertices in R_n) **We note that for any set of patterns σ , the similar as above happens in case of all $R_n(\sigma)$'s, varying n upto infinity.**

We can extend those trees as a **lattice** in a natural way **tagging a vertex ϵ in the bottom of those trees** and connecting ϵ with the bottom vertices of the tree by an edge respectively. We denote the corresponding lattices from the pattern avoidance classes as $R_{n,L}(\sigma)$. We note that many similar things happen for $R_{n,L}$ and $R_{n,L}(\sigma)$ while the lattice structure is imposed in a natural way via ϵ .

Note: [2], (Ref P-2) **Every monotone epimorphism is confluent.**

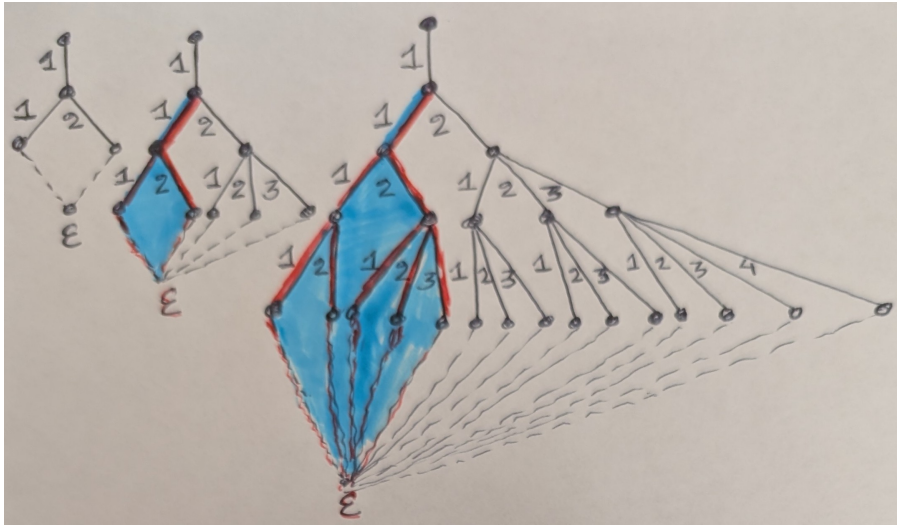


Figure: $R_{2,L} \subseteq R_{3,L} \subseteq R_{4,L}$ (as highlighted in the bottom left)

As in [2] (Ref P-2) Given sequences F_n , of finite graphs and epimorphisms $\alpha_n: F_{n+1} \mapsto F_n$, we denote the inverse sequence by (F_n, α_n) .

Given an inverse sequence (F_n, α_n) , the associated inverse limit space is the subspace of the product space $\prod F_i$ determined by $\{(x_1, x_2, \dots) : x_i \in F_i, x_i = \alpha_i(x_{i+1})\}$ and is denoted as $F = \varprojlim (F_n, \alpha_n)$.

If $a = (a_n), b = (b_n)$ are elements of F , then $\langle a, b \rangle$ is an **edge** in F if and only if, for each n $\langle a_n, b_n \rangle$ is an edge in F_n .

Lemma

[2], (Ref P-2) If $\mathcal{T}$ is a Projective Fraisse Family of *rooted trees* and if T is the Projective Fraisse limit, then T is arcwise connected.

Thus we note that

1. If $R = \varprojlim R_n$, then R is arcwise connected.
2. If $R(\sigma) = \varprojlim R_n(\sigma)$, then $R(\sigma)$ is arcwise connected.

Definition

Definition 6.3, [2], (Ref P-2) A topological graph X is called **Kelley** if X is connected and for every closed and connected set $K \subseteq X$, every vertex $p \in K$, and every sequence (p_n) converging to p of vertices in X there are closed and connected sets K_n such that $p_n \in K_n$ and $\varprojlim K_n = K$.

Lemma

*Proposition 6.5, [2], (Ref P-2) If G is a **Projective Fraisse family of connected finite graphs** with confluent epimorphisms, then the **Projective Fraisse Limit is Kelley**.*

EXAMPLES from Pattern avoidance: Consider $R_{n,k}(1/23)$, which is the collection of all words $w \in R_{n,k}$, where w is obtained by inserting a single 1 into a word of the form $1^l 234 \dots k$, for some $l \geq 0, k \geq 1$ following [3], [4]: We note that

1. The Hasse diagram of $R_{n,k+1}(1/23)$ contains a copy of that of $R_{n,k}(1/23)$, where $1 \leq k < n$
2. The Hasse diagram of $R_{n+1,k}(1/23)$ contains a copy of that of $R_{n,k}(1/23)$ leaving the top edge, (where $1 \leq k < n$)

Following these observations, we can create an epimorphism of graphs as $\phi_{n,k+1} : R_{n,k+1}(1/23) \rightarrow R_{n,k}(1/23)$ and $\phi_{n+1,k} : R_{n+1,k}(1/23) \rightarrow R_{n,k}(1/23)$ and can compose the epimorphisms accordingly.

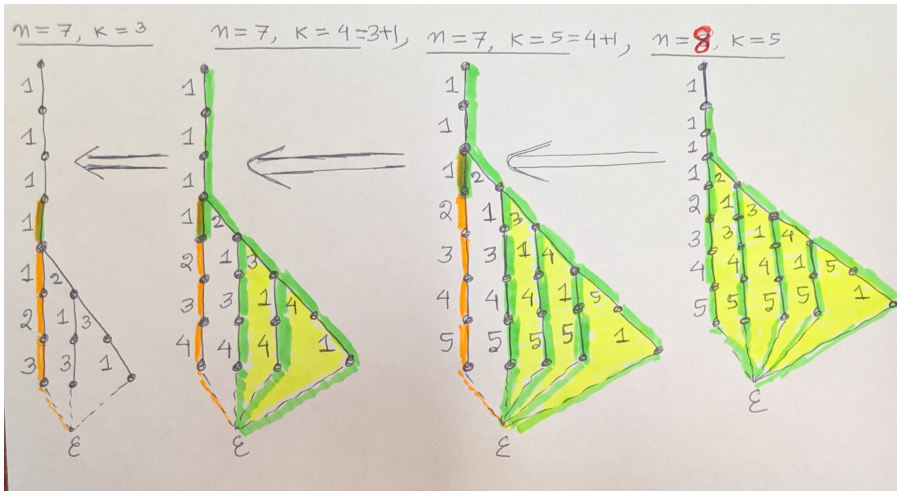


Figure: $R_{7,3,L} \subseteq R_{7,4,L} \subseteq R_{7,5,L} \subseteq R_{8,5,L}$ (as highlighted in the right and bottom right)

Thus we can create a Projective Fraisse family of trees with **confluent** (as in definition 4.1 in [2])(Ref P-2) epimorphisms.

Extending the bonding maps in the corresponding **lattice** structures varying **both n and k** in a natural way, we get a **Projective Fraisse family of graphs (from the Hasse diagrams) with confluent epimorphisms.**

So, by [2], (Ref P-2) the corresponding **projective limit, while both n, k vary, is kelley** (in case of tree and lattice both), while the projective limit of the **trees is arcwise connected.**

Definition

[3] (Ref P-2) A standard type B Partition of

$\langle n \rangle = \{-n, \dots, -1, 0, 1, \dots, n\}$ of the form $\rho = S_0/S_1/S_2/\dots/S_k$ satisfying

1. $0 \in S_0$, and if $i \in S_0$, then $\bar{i} \in S_0$, where $\bar{i} = -i$
2. For $i \geq 1$, we have $S_{2i} = -S_{2i-1}$, where $-S = \{-s : s \in S\}$
3. $m_{2i} \in S_{2i}, \forall i$, where $m_i = \min|S_i|$, with $|S| = \{s : s \in S\}$
4. $0 = m_0 < m_2 < m_4 < \dots m_{2k}$

Definition

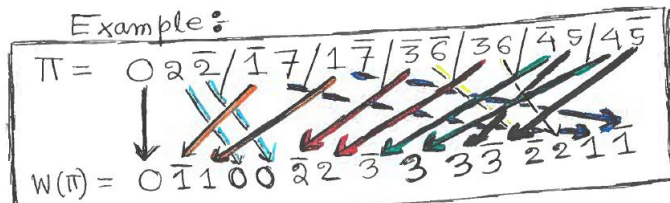
[4] (Ref P-2) A signed restricted growth function (S.R.G.F.) is a sequence of the form $w = a_0 a_1 a_1^* a_2 a_2^* \cdots a_n a_n^*$ of length $2n + 1$, where

1. $a_0 = 0$
2. If $a_i = j$, then $a_i^* = \bar{j}$ and conversely $\forall i, j \in \{1, 2, \dots, n\}$
3. $\forall i \geq 1, |a_{i+1}| \leq 1 + \text{Max}\{|a_0|, |a_1| \cdots |a_{i-1}|\}$
4. The pair $\bar{j}j$ appears before $j\bar{j}$ (if there is a $j\bar{j}$ in the sequence, for any $j \in \{1, 2, 3, \dots, n\}$).

We find analogous bijection in between $S_B(\langle n \rangle, k)$ (The set of all standard type B partitions of $\langle n \rangle$ into $2k + 1$ many blocks) and $SR_{n,k}$, that is (S.R.G.F.) the set of all signed restricted growth functions of length $2n + 1$, with largest positive integer digit k as follows:

[4] (Ref P-2) Consider any $\Pi \in \mathcal{S}_B(\langle n \rangle, k)$. Associate with Π , the word $w(\Pi) = a_0 a_1 a_1^* a_2 a_2^* \cdots a_n a_n^*$, where

1. $a_0 = 0, a_i = a_i^* = 0$, iff $i, \bar{i} \in S_0$, for $i \in \{1, 2, \dots, n\}$
2. $a_i = \bar{j}, a_i^* = j$, iff $i \in S_{2j}$, for $i, j \in \{1, 2, \dots, n\}$
3. $a_i = j, a_i^* = \bar{j}$, iff $i \in S_{2j-1}$, for $i, j \in \{1, 2, \dots, n\}$



EXAMPLE from Type B partitions: [1]

Labeling the edges of **maximal chains via s.r.g.f.** in a similar way, we observe that the corresponding lattice $R_{n+1,k,L}^B$ contains a copy of $R_{n,k,L}^B$ in its **bottom right corner**. We can define a **confluent epimorphism of graphs** from $R_{n+1,k,L}^B$ onto $R_{n,k,L}^B$ in a similar way as in the previous example. If we **ignore the lattice structure, omitting the ' ϵ '** and the edges connecting to ϵ , then we see again the corresponding projective limit (where n varies upto infinity for a fixed k) is **arcwise connected, and Kelley**, whereas the projective limit of the **corresponding lattice is Kelley[2](Ref: P-2)**.

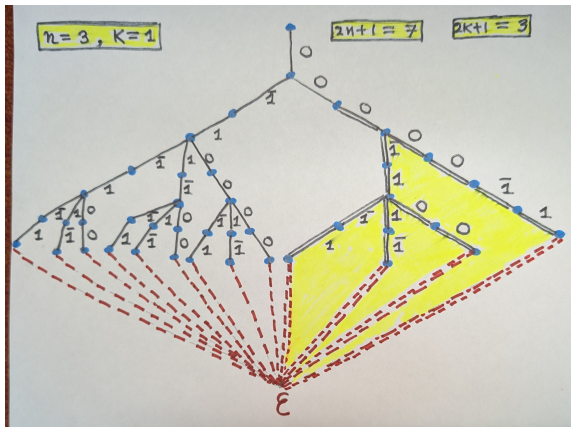


Figure: $R_{3,1,L}^B$ containing $R_{2,1,L}^B$ (as highlighted)

EXAMPLES from non crossing partitions:[1]

1st example: We notice that if $NC(n, k)$ is the set of all non crossing partitions of $[n]$ in to k many blocks, then for $k = 2$, always

the lattice of $NC(n, 2)$ contains a copy of that of $NC(n - 1, 2)$

Hence, we see a natural epimorphism from the Hasse diagram of

$NC(n, 2)$ onto that of $NC(n - 1, 2)$, by utilising the technique of sequence of first occurrence $F(\rho)$ and it's complement $R(\rho)$ for any

r.g.f. ρ as in [6](Ref P-2). By proposition 6.5 in [2] (Ref P-2), the corresponding

projective limit N is Kelley, since the bonding epimorphisms are monotone (having each vertex fiber connected).

Note: In these poset we don't label the edges as previously, we rather label the vertex.

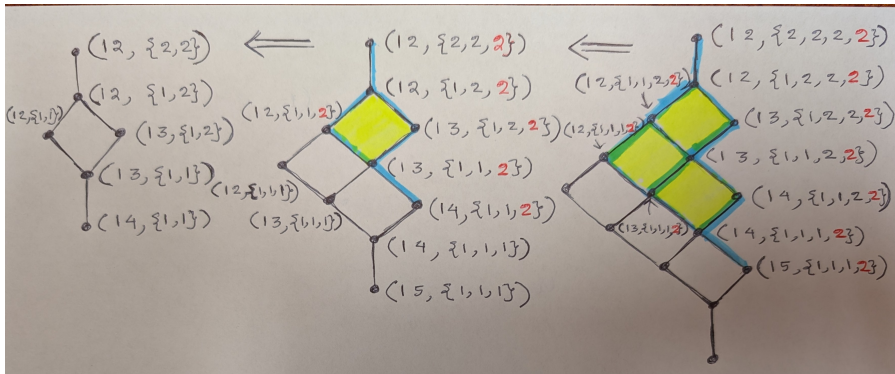
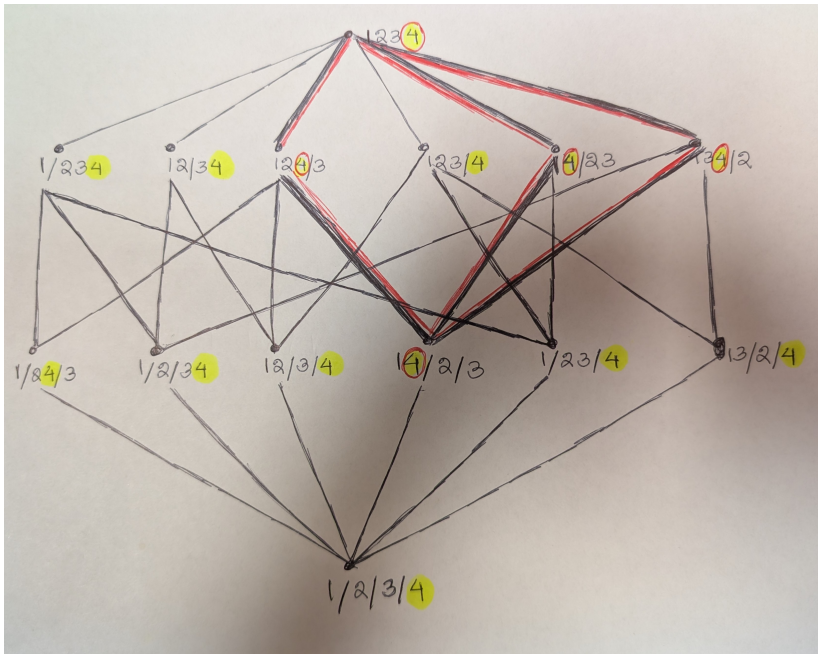


Figure: The lattices of $NC(4, 2) \subseteq NC(5, 2) \subseteq NC(6, 2)$ (as highlighted)

2nd example: Here we do not use r.g.f. We rather use the usual partial orders of set partitions to draw the hass diagrams, where for $\pi, \sigma \in \Pi_n, \pi \leq \sigma$, iff each block of π is contained in some block of σ . We note that the Hass diagram of NC_{n+1} (The non crossing set partitions of $[n+1]$) contains that of NC_n . Hence $\exists$ a natural epimorphism $\phi_n : NC_{n+1} \mapsto NC_n$, defined as $\phi_n(\alpha) = \alpha'$, where α' is the non crossing partition of $[n]$ obtained from α by deleting the $n+1$. We notice that, each ϕ_n is a epimorphism of graphs preserving the natural partial order of set partitions and the set of all the NC_n 's along with the ϕ_n 's form a Projective Fraisse Family. Note that Φ_n is not monotone, since for example $\phi_4^{-1}(1/2/3) = \{14/2/3, 1/24/3, 1/2/34\}$ is not connected.



Questions for future works:

1. Can we form such Projective Fraisse family of trees and lattices varying both n and k for standard type B partitions (with $1 < k < n$)?
2. Can we form Projective Fraisse family of lattices varying n upto infinity and fixing any k in case of non crossing partitions (and more generally varying both n and k upto infinity with $1 < k < n$)?
3. Are the above bonding epimorphisms ϕ_n (or there may exist possibly other bonding epimorphisms) from NC_{n+1} onto NC_n confluent making the Projective limit Kelley?

References



Amrita Acharyya, *Inverse limits of various posets* , Arxiv:
2512.12007v1



Rodica Simion and Frank W. Schmidt, *Restricted Permutations*,
Europ. J. Combinatorics (1985)6, 383-406



S Dahlberg, R Dorward, J Gerhard, T Grubb, C Purcell, L
Reppuhn, B. E. Sagan, *Set partition patterns and statistics*,
Discrete Mathematics 339 (2016) 1–16



Amrita Acharyya, Robinson Paul Czajkowski, Allen Richard
Williams *K block Set Partition Pattern And Statistics* Preprint
arxiv:2003.02915

-  W.J.Chartonik, A Kwiakowska, R.P.Roe, *The Projective Fraisse Limit of the family of all connected finite graphs with confluent epimorphisms*, Arxiv:2206.12400V3
-  W.J.Chartonik, A Kwiakowska, R.P.Roe, S. Yang *Projective Fraisse Limit of trees with confluent epimorphisms*, Arxiv:2312.16915V2
-  Bruce E. Sagan, Joshua P. Swanson *q-Stirling numbers in type B* European Journal of Combinatorics, Vol 118, 2024
-  Amrita Acharyya *Type B set partitions, an analogue of restricted growth functions*, Ars Combinatoria 162 (2025) 191-204
-  E. Bagno, D. Garber *Combinatorics of q-r-analogues of Stirling numbers of Type B*, Arxiv: 2401.08365
-  Jinting Liang, Bruce E. Sagan *Log-concavity and log-convexity via distributive lattices*, Arxiv:2408.02782v1

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